DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCES- MATHEMATICS

SEMESTER -IV

ABSTRACT MEASURE THEORY

DEMATH4CORE1

BLOCK-1

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FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours

ABSTRACT MEASURE THEORY

BLOCK-1

BLOCK-2

INTRODUCTION TO BLOCK-I

This block discusses about σ-algebra, its monotone classes, its restrictions and about Borel σ-algebra.we study about general measures, Point mass distributions, complete measures, restrictions and its uniqueness. We discusses different kinds of borel measures, outer measures and its constructions ,volume of intervals , lebesgue measure and its transformations and also about cantor set,cantor ternary set and its functions,different functions and arithmetic operations which we can perform on the measurable functions.

In this block We will be learning about the devil's staircase and seeing problems related to it.

UNIT 1 Σ-ALGEBRAS

STRUCTURE

1.1 OBJECTIVES

In this unit we are going to learn about σ-algebra, its monotone classes, its restrictions and about Borel σ-algebra.

1.2 INTRODUCTION

In **mathematical** analysis and in probability theory, a **σ-algebra** (also **σ-field**) on a set X is a collection **Σ** of subsets of X that includes X itself, is closed under complement, and is closed under countable unions. ... Also, in probability, **σ-algebras** are pivotal in the definition of conditional expectation.

Definition 1.1*Let X be a non-empty set and Σ a collection of subsets of X. We call Σ a σ-algebra of subsets of X if it is non-empty, closed under complements and closed under countable unions. This means:*

(i) there exists at least one $A \subseteq X$ so that $A \in \Sigma$,

(ii) if $A \in \Sigma$, then $A^c \in \Sigma$, where $A^c = X \setminus A$, and (iii) if $A_n \in \Sigma$ for all $n \in N$, then $\cup_{n=1}^{+\infty} A_n \in \Sigma$.

The pair (X,Σ) of a non-empty set X and a σ-algebra Σ of subsets of X is called a measurable space.

Proposition 1.1 *Every σ-algebra of subsets of X contains at least the sets ∅ and X, it is closed under finite unions, under countable intersections, under finite intersections and under set-theoretic differences.*

Proof: Let Σ be any σ-algebra of subsets of X.

(a) Take any $A \in \Sigma$ and consider the sets $A_1 = A$ and $A_n = A^c$ for all $n \geq 2$. Then$X = A \cup A^c = \cup_{n=1}^{+\infty} A_n \in \Sigma$ and also $\emptyset = X^c \in \Sigma$.

(b) Let $A_1,...,A_N \in \Sigma$. Consider $A_n = A_N$ for all $n > N$ and get that $\cup_{n=1}^{N} A_n = \cup_{n=1}^{+\infty} A_n \in \Sigma$.

(c) Let $A_n \in \Sigma$ for all n. Then$\cap_{n=1}^{+\infty} A_n = (\cup_{n=1}^{+\infty} A_n^c)^c \in \Sigma$.

(d) Let $A_1,...,A_N \in \Sigma$. Using the result of (b), we get that $\cap_{n=1}^{N} A_n = (\cup_{n=1}^{N} A_n^c)^c \in \Sigma$.

(e) Finally, let $A,B \in \Sigma$. Using the result of (d), we get that $A \setminus B = A \cap B^c \in \Sigma$.

Here are some simple examples.

Examples

1. The collection $\{\emptyset, X\}$ is a σ-algebra of subsets of X.
2. If $E \subseteq X$ is non-empty and different from X, then the collection $\{\emptyset, E, E^c, X\}$ is a σ-algebra of subsets of X.

3.P(X), the collection of all subsets of X, is a σ-algebra of subsets of X.

3. Let X be uncountable. The $\{A \subseteq X \mid A$ is countable or A^c is countable$\}$ is a σ-algebra of subsets of X. Firstly, ∅ is countable and, hence, the collection is non-empty. If A is in the collection, then, considering cases, we see that A^c is also in the collection. Finally, let A_n be in the collection for all $n \in \mathbf{N}$. If all A_n's are countable, then$\cup_{n=1}^{+\infty} A_n$ is also countable. If at least one of theA_n^c's, say$A_{n_0}^c$, is countable, then

$(\cup_{n=1}^{+\infty} A_n)^c \subseteq A_{n_0}^c$ is also countable. In any case, $\cup_{n=1}^{+\infty} A_n$ belongs to the collection.

The following result is useful.

Proposition 1.2 *Let* Σ *be a σ-algebra of subsets of X and consider a finite sequence* $\{A_n\}_{n=1}^N$ *or an infinite sequence* $\{A_n\}$ *in* Σ. *Then there exists a finite sequence* $\{B_n\}_{n=1}^N$ *or, respectively, an infinite sequence* $\{B_n\}$ *in* Σ *with the properties:*

(i) $B_n \subseteq A_n$ *for all* $n = 1,\dots,N$ *or, respectively, all* $n \in \mathbf{N}$.

(ii) $\cup_{n=1}^N B_n = \cup_{n=1}^N A_n$ *or, respectively,* $\cup_{n=1}^{+\infty} B_n = \cup_{n=1}^{+\infty} A_n$.

(iii) the B_n*'s are pairwise disjoint.*

Proof: Trivial, by taking $B_1 = A_1$ and $B_k = A_k \setminus (A_1 \cup \cdots \cup A_{k-1})$ for all $k = 2,\dots,N$ or, respectively, all $k = 2,3,\dots$.

1.3 GENERATED Σ-ALGEBRAS.

Proposition 1.3 *The intersection of any σ-algebras of subsets of the same X is a σ-algebra of subsets of X.*

Proof: Let $\{\Sigma_i\}_{i\in I}$ be any collection of σ-algebras of subsets of X, indexed by an arbitrary non-empty set I of indices, and consider the intersection $\Sigma = \cap_{i\in I}\Sigma_i$. Since $\emptyset \in \Sigma_i$ for all $i \in I$, we get $\emptyset \in \Sigma$ and, hence, Σ is non-empty.

Let $A \in \Sigma$. Then $A \in \Sigma_i$ for all $i \in I$ and, since all Σ_i's are σ-algebras, $A^c \in \Sigma_i$ for all $i \in I$. Therefore $A^c \in \Sigma$.

Let $A_n \in \Sigma$ for all $n \in \mathbf{N}$. Then $A_n \in \Sigma_i$ for all $i \in I$ and all $n \in \mathbf{N}$ and, since all Σ_i's are σ-algebras, we get $\cup_{n=1}^{+\infty} A_n \in \Sigma_i$ for all $i \in I$. Thus, $\cup_{n=1}^{+\infty} A_n \in \Sigma$.

Definition 1.2 *Let X be a non-empty set and* E *be an arbitrary collection of subsets of X. The intersection of all σ-algebras of subsets of X which include* E *is called the σ-algebra generated by* E *and it is denoted by* Σ(E). *Namely*

$$\Sigma(\mathrm{E}) = \cap\{\Sigma \mid \Sigma \text{ is a } \sigma\text{-algebra of subsets of } X \text{ and } \mathrm{E} \subseteq \Sigma\}.$$

Note that there is at least one σ-algebra of subsets of X which includes E and this is P(X). Note also that the term *σ-algebra* used in the name of Σ(E) is justified by its definition and by Proposition 1.3.

Proposition 1.4 *Let* E *be any collection of subsets of the non-empty X. Then* Σ(E) *is the smallest σ-algebra of subsets of X which includes* E. *Namely, if* Σ *is any σ-algebra of subsets of X such that* E $\subseteq \Sigma$, *then* Σ(E) $\subseteq \Sigma$.

Proof: If Σ is any σ-algebra of subsets of X such that E $\subseteq \Sigma$, then Σ is one of the σ-algebras whose intersection is denoted Σ(E). Therefore Σ(E) $\subseteq$ Σ.

Looking back at two of the examples of σ-algebras, we easily get the following examples.

Examples.

1. Let $E \subseteq X$ and E be non-empty and different from X and consider E = $\{E\}$. Then Σ(E) = $\{\emptyset, E, E^c, X\}$. To see this just observe that $\{\emptyset, E, E^c, X\}$ is a σ-algebra of subsets of X which contains E and that there can be no smaller σ-algebra of subsets of X containing E, since such a σ-algebra must necessarily contain $\emptyset$, X and E^c besides E.
2. Let X be an uncountable set and consider E = $\{A \subseteq X \mid A$ is countable$\}$. Then Σ(E) = $\{A \subseteq X \mid A$ is countable or A^c is countable$\}$. The argument is the same as before. $\{A \subseteq X \mid A$ is countable or A^c is countable$\}$ is a σ-algebra of subsets of X which contains all countable subsets of X and there is no smaller σ-algebra of subsets of X containing all countable subsets of X, since any such σ-algebra must contain all the complements of countable subsets of X.

1.4 ALGEBRAS AND MONOTONE CLASSES.

Definition 1.3 *Let X be non-empty and* A *a collection of subsets of X. We call* A *an algebra of subsets of X if it is non-empty, closed under complements and closed under unions. This means:*

(i) there exists at least one $A \subseteq X$ so that $A \in$ A,

(ii) if $A \in$ A*, then* $A^c \in$ A *and (iii) if* $A,B \in$ A*, then* $A \cup B \in$ A.

Proposition 1.5 *Every algebra of subsets of X contains at least the sets* $\emptyset$ *and X, it is closed under finite unions, under finite intersections and under set-theoretic differences.*

Proof: Let A be any algebra of subsets of X.

(a) Take any $A \in$ A and consider the sets A and A^c. Then $X = A \cup A^c \in$ A and then $\emptyset = X^c \in$ A.

(b) It is trivial to prove by induction that for any $n \in \mathbf{N}$ and any $A_1,...,A_n \in$ A it follows $A_1 \cup \cdots \cup A_n \in$ A.

(c) By the result of (b), if $A_1,...,A_n \in$ A, then $\cap_{k=1}^{n} A_k = (\cup_{k=1}^{n} A_k^c)^c \in \mathcal{A}$.

(d) If $A,B \in$ A, using the result of (c), we get that $A \setminus B = A \cap B^c \in$ A.

Examples.

1. Every σ-algebra is also an algebra.
2. If X is an infinite set then the collection $\{A \subseteq X \mid A$ is finite or A^c is finite$\}$ is an algebra of subsets of X.

If (A_n) is a sequence of subsets of a set X and $A_n \subseteq A_{n+1}$ for all n, we say that the sequence is *increasing*. In this case, if $A = \cup_{n=1}^{+\infty} A_n$, we write

$$A_n \uparrow A.$$

If $A_{n+1} \subseteq A_n$ for all n, we say that the sequence (A_n) is *decreasing* and, if also $A = \cap_{n=1}^{+\infty} A_n$, we write

$$A_n \downarrow A.$$

1.5 RESTRICTION OF A *Σ*-ALGEBRAS.

Proposition 1.8 *Let* Σ *be a* σ*-algebra of subsets of* X *and* $Y \subseteq X$ *be nonempty. If we denote*

$$\Sigma e Y = \{A \cap Y \mid A \in \Sigma\},$$

then $\Sigma e Y$ *is a* σ*-algebra of subsets of* Y *.*

In case $Y \in \Sigma$*, we have* $\Sigma e Y = \{A \subseteq Y \mid A \in \Sigma\}$.

Proof: Since $\emptyset \in \Sigma$, we have that $\emptyset = \emptyset \cap Y \in \Sigma e Y$.

If $B \in \Sigma e Y$, then $B = A \cap Y$ for some $A \in \Sigma$. Since $X \setminus A \in \Sigma$, we get that $Y \setminus B = (X \setminus A) \cap Y \in \Sigma e Y$.

If $B_1, B_2, \ldots \in \Sigma\rceil Y$, then, for each k, $B_k = A_k \cap Y$ for some $A_k \in \Sigma$. Since $\cup_{k=1}^{+\infty} A_k \in \Sigma$, we find that $\cup_{k=1}^{+\infty} B_k = (\cup_{k=1}^{+\infty} A_k) \cap Y \in \Sigma\rceil Y$.

Now let $Y \in \Sigma$. If $B \in \Sigma\rceil Y$, then $B = A \cap Y$ for some $A \in \Sigma$ and, hence, $B \subseteq Y$ and $B \in \Sigma$. Therefore $B \in \{C \subseteq Y \mid C \in \Sigma\}$. Conversely, if $B \in \{C \subseteq Y \mid C \in \Sigma\}$, then $B \subseteq Y$ and $B \in \Sigma$. We set $A = B$ and, thus, $B = A \cap Y$ and $A \in \Sigma$. Therefore $B \in \Sigma\rceil Y$.

Definition 1.6 *Let Σ be a σ-algebra of subsets of X and let $Y \subseteq X$ be nonempty. The σ-algebra $\Sigma\rceil Y$, defined in Proposition 1.8, is called the restriction of Σ on Y.*

In general, if $\mathcal{E}$ is any collection of subsets of X and $Y \subseteq X$, we denote

$$\mathcal{E}\rceil Y = \{A \cap Y \mid A \in \mathcal{E}\}$$

and call $\mathcal{E}\rceil Y$ **the restriction of** $\mathcal{E}$ **on** Y.

Theorem 1.2 *Let $\mathcal{E}$ be a collection of subsets of X and $Y \subseteq X$ be non-empty.*

Then

$$\Sigma(\mathcal{E}\rceil Y) = \Sigma(\mathcal{E})\rceil Y,$$

where $\Sigma(\mathcal{E}\rceil Y)$ is the σ-algebra of subsets of Y generated by $\mathcal{E}\rceil Y$.

Proof: If $B \in \mathcal{E}\rceil Y$, then $B = A \cap Y$ for some $A \in \mathcal{E} \subseteq \Sigma(\mathcal{E})$ and, thus, $B \in \Sigma(\mathcal{E})\rceil Y$. Hence, $\mathcal{E}\rceil Y \subseteq \Sigma(\mathcal{E})\rceil Y$ and, since, by Proposition 1.8, $\Sigma(\mathcal{E})\rceil Y$ is a σ-algebra of subsets of Y, Proposition 1.4 implies $\Sigma(\mathcal{E}\rceil Y) \subseteq \Sigma(\mathcal{E})\rceil Y$. Now, define the collection

$$\Sigma = \{A \subseteq X \mid A \cap Y \in \Sigma(\mathcal{E}\rceil Y)\}.$$

We have that $\emptyset \in \Sigma$, because $\emptyset \cap Y = \emptyset \in \Sigma(\mathcal{E}\rceil Y)$.

If $A \in \Sigma$, then $A \cap Y \in \Sigma(\mathcal{E}\rceil Y)$. Therefore, $X \setminus A \in \Sigma$, because $(X \setminus A) \cap Y = Y \setminus (A \cap Y) \in \Sigma(\mathcal{E}\rceil Y)$.

If $A_1, A_2, \ldots \in \Sigma$, then $A_1 \cap Y, A_2 \cap Y, \ldots \in \Sigma(\mathcal{E}\rceil Y)$. This implies that $(\cup_{k=1}^{+\infty} A_k) \cap Y = \cup_{k=1}^{+\infty} (A_k \cap Y) \in \Sigma(\mathcal{E}\rceil Y)$ and, thus, $\cup_{k=1}^{+\infty} A_k \in \Sigma$. We conclude that Σ is a σ-algebra of subsets of X.

If $A \in \mathcal{E}$, then $A \cap Y \in \mathcal{E}\rceil Y \subseteq \Sigma(\mathcal{E}\rceil Y)$ and, hence, $A \in \Sigma$. Therefore, $\mathcal{E} \subseteq \Sigma$ and, by Proposition 1.4, $\Sigma(\mathcal{E}) \subseteq \Sigma$. Now, for an arbitrary $B \in \Sigma(\mathcal{E})\rceil Y$, we have that $B = A \cap Y$ for some $A \in \Sigma(\mathcal{E}) \subseteq \Sigma$ and, thus, $B \in \Sigma(\mathcal{E}\rceil Y)$. This implies that $\Sigma(\mathcal{E})\rceil Y \subseteq \Sigma(\mathcal{E}\rceil Y)$.

1.6 BOREL Σ-ALGEBRAS.

Definition 1.7 *Let X be a topological space and* T *the topology of X, i.e. the collection of all open subsets of X. The σ-algebra of subsets of X which is generated by* T *, namely the smallest σ-algebra of subsets of X containing all open subsets of X, is called the Borel σ-algebra of X and we denote it* B_X*:* $B_X = \Sigma(T)$, T *the topology of X.*

The elements of B_X *are called Borel sets in X and* B_X *is also called the σ-algebra of Borel sets in X.*

By definition, all open subsets of X are Borel sets in X and, since B_X is a σ-algebra, all closed subsets of X (which are the complements of open subsets) are also Borel sets in X. A subset of X is called a G_δ**-set** if it is a countable intersection of open subsets of X. Also, a subset of X is called an F_σ**-set** if it is a countable union of closed subsets of X. It is obvious that all G_δ-sets and all F_σ-sets are Borel sets in X.

Proposition 1.9 *If X is a topological space and* F *is the collection of all closed subsets of X, then* $B_X = \Sigma(F)$.

Proof: Every closed set is contained in $\Sigma(T)$. This is true because $\Sigma(T)$ contains all open sets and hence, being a σ-algebra, contains all closed sets. Therefore, $F \subseteq \Sigma(T)$. Since $\Sigma(T)$ is a σ-algebra, Proposition 1.4 implies $\Sigma(F) \subseteq \Sigma(T)$.

Symmetrically, every open set is contained in $\Sigma(F)$. This is because $\Sigma(F)$ contains all closed sets and hence, being a σ-algebra, contains all open sets (the complements of closed sets). Therefore, $T \subseteq \Sigma(F)$. Since $\Sigma(F)$ is a σ-algebra, Proposition 1.4 implies $\Sigma(T) \subseteq \Sigma(F)$. Therefore, $\Sigma(F) = \Sigma(T) = B_X$.

If X is a topological space with the topology T and if $Y \subseteq X$, then, as is wellknown (and easy to prove), the collection T eY = $\{U \cap Y \mid U \in T\}$ is a topology of Y which is called **the relative topology** or **the subspace topology** of Y .

Theorem 1.3 *Let X be a topological space and let the non-empty* $Y \subseteq X$ *have the subspace topology. Then*

$$B_Y = B_X \text{e} Y.$$

Proof: If T is the topology of X, then T eY is the subspace topology of Y. Theorem 1.2 implies that $\mathrm{B}_Y = \Sigma(\mathrm{T}\ \mathrm{e}Y) = \Sigma(\mathrm{T})\mathrm{e}Y = \mathrm{B}_X\mathrm{e}Y$.

Thus, *the Borel sets in the subset Y of X (with the subspace topology of Y) are just the intersections with Y of the Borel sets in X.*

Examples of topological spaces are all metric spaces of which the most familiar is the euclidean space $X = \mathbf{R}^n$ with the usual euclidean metric or even any subset X of $\mathbf{R}^n$ with the restriction on X of the euclidean metric. Because of the importance of $\mathbf{R}^n$ we shall pay particular attention on $\mathrm{B}_{\mathbf{R}^n}$.

Lemma 1.1 *All n-dimensional intervals are Borel sets in* $\mathbf{R}^n$.

Proof: For any $j = 1,...,n$, a half-space of the form $\{x = (x_1,...,x_n)|x_j < b_j\}$ or of the form $\{x = (x_1,...,x_n)|x_j \le b_j\}$ is a Borel set in $\mathbf{R}^n$, since it is an open set in the first case and a closed set in the second case. Similarly, a half-space of the form $\{x = (x_1,...,x_n)|a_j < x_j\}$ or of the form $\{x = (x_1,...,x_n)|a_j \le x_j\}$ is a Borel set in $\mathbf{R}^n$. Now, every interval S is an intersection of $2n$ of these half-spaces and, therefore, it is also a Borel set in $\mathbf{R}^n$.

Proposition 1.10 *If* E *is the collection of all closed or of all open or of all open-closed or of all closed-open or of all intervals in* $\mathbf{R}^n$, *then* $\mathrm{B}_{\mathbf{R}^n} = \Sigma(\mathrm{E})$.

Proof: By Lemma 1.1 we have that, in all cases, $\mathrm{E} \subseteq \mathrm{B}_{\mathbf{R}^n}$. Proposition 1.4 implies that $\Sigma(\mathrm{E}) \subseteq \mathrm{B}_{\mathbf{R}^n}$.

To show the opposite inclusion consider any open subset U of $\mathbf{R}^n$. For every $x \in U$ find a small open ball B_x centered at x which is included in U. Now, considering the case of E being the collection of all closed intervals, take an arbitrary $Q_x = [a_1,b_1] \times \cdots \times [a_n,b_n]$ containing x, small enough so that it is included in B_x, and hence in U, and with all $a_1,...,a_n,b_1,...,b_n$ being *rational* numbers. Since $x \in Q_x \subseteq U$ for all $x \in U$, we have that $U = \cup_{x\in U}Q_x$. But the collection of all possible Q_x's is countable (!) and, thus, the general open subset U of $\mathbf{R}^n$ can be written as a countable union of sets in the collection E. Hence every open U belongs to $\Sigma(\mathrm{E})$ and, since $\Sigma(\mathrm{E})$ is a σ-algebra of subsets of $\mathbf{R}^n$ and $\mathrm{B}_{\mathbf{R}^n}$ is generated by the collection of all open subsets of $\mathbf{R}^n$, Proposition 1.4 implies that $\mathcal{B}_{\mathbf{R}^n} \subseteq \Sigma(\mathcal{E})$.

Of course, the proof of the last inclusion works in the same way with all other types of intervals.

As we said, the intervals in $\mathbf{R}^n$ are cartesian products of *n bounded* intervals in **R**. If we allow these intervals in **R** to become unbounded, we get the so-called **generalized intervals** in $\mathbf{R}^n$, namely all sets of the form $I_1 \times \cdots \times I_n$, where each I_j is any, even unbounded, interval in **R**.

Check your progress

1.Let A be an algebra of subsets of X. Prove that A is a σ-algebra if and only if it is closed under increasing countable unions.

2. Let X be non-empty. In the next three cases find $\Sigma(\mathcal{E})$ and $\mathcal{M}(\mathcal{E})$. (i) $\mathcal{E} = \emptyset$.

(ii) Fix $E \subseteq X$ and let $\mathcal{E} = \{F \mid E \subseteq F \subseteq X\}$.

(iii) Let $\mathcal{E} = \{F \mid F$ is a two-point subset of $X\}$.

3. Let $\mathcal{E}_1, \mathcal{E}_2$ be two collections of subsets of the non-empty X. If $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \Sigma(\mathcal{E}_1)$, prove that $\Sigma(\mathcal{E}_1) = \Sigma(\mathcal{E}_2)$.

4. Let Y be a non-empty subset of X.

(i) If A is an algebra of subsets of X, prove that A$\text{e}Y$ is an algebra of subsets of Y .

(ii) If M is a monotone class of subsets of X, prove that M$\text{e}Y$ is a monotone class of subsets of Y .

(iii) If T is a topology of X, prove that T $\text{e}Y$ is a topology of Y .

Let X be a topological space and Y be a non-empty Borel set in X. Prove that $\mathcal{B}_Y = \{A \subseteq Y \mid A \in \mathcal{B}_X\}$.

1.7 LET US SUM UP

In this Unit we discussed the following points

- Borel σ-algebras
- Restriction of a σ-algebras
- Algebras and monotone classes

- Generated σ-algebras.

1.8 KEYWORDS

Algebra-the part of mathematics in which letters and other general symbols are used to represent numbers and quantities in formulae and equations.

Monotone class-**monotone class** is a **class** M of sets that is closed under countable **monotone**unions and intersections, i.e. if and then. , and similarly in the other direction.

1.9. QUESTIONS FOR REVIEW

1. Let X be a non-empty set and $A_1, A_2, \ldots \subseteq X$. We define

$\limsup_{n\to+\infty} A_n = \cap_{k=1}^{+\infty}\left(\cup_{j=k}^{+\infty} A_j\right)$, $\liminf_{n\to+\infty} A_n = \cup_{k=1}^{+\infty}\left(\cap_{j=k}^{+\infty} A_j\right)$.

Only in case $\liminf_{n\to+\infty} A_n = \limsup_{n\to+\infty} A_n$, we define

$\lim A_n = \liminf A_n = \limsup A_n$. $n\to+\infty$ $n\to+\infty$ $n\to+\infty$

Prove the following.

(i) $\limsup_{n\to+\infty} A_n = \{x \in X \mid x \in A_n$ for infinitely many $n\}$.

(ii) $\liminf_{n\to+\infty} A_n = \{x \in X \mid x \in A_n$ for all large enough $n\}$.

(iii) $(\liminf_{n\to+\infty} A_n)^c = \limsup_{n\to+\infty} A_n^c$ and $(\limsup_{n\to+\infty} A_n)^c = \liminf_{n\to+\infty} A_n^c$.

(iv) $\liminf n\to+\infty\ An \subseteq \limsup n\to+\infty\ An$.

(v) If $A_n \subseteq A_{n+1}$ for all n, then $\lim_{n\to+\infty} A_n = \cup_{n=1}^{+\infty} A_n$.

(vi) If $A_{n+1} \subseteq A_n$ for all n, then $\lim_{n\to+\infty} A_n = \cap_{n=1}^{+\infty} A_n$.

(vii) Find an example where $\liminf_{n\to+\infty} A_n$ 6= $\limsup_{n\to+\infty} A_n$. (viii) If $A_n \subseteq B_n$ for all n, then $\limsup_{n\to+\infty} A_n \subseteq \limsup_{n\to+\infty} B_n$ and $\liminf n\to+\infty\ An \subseteq \liminf n\to+\infty\ Bn$.

(ix) If $A_n = B_n \cup C_n$ for all n, then $\limsup_{n\to+\infty} A_n \subseteq \limsup_{n\to+\infty} B_n \cup$ $\limsup n\to+\infty\ Cn$, $\liminf n\to+\infty\ Bn \cup \liminf n\to+\infty\ Cn \subseteq \liminf n\to+\infty\ An$.

2.Push-forward of a σ-algebra.

Let Σ be a σ-algebra of subsets of X and let $f : X \to Y$. Then the collection

$$\{B \subseteq Y \mid f^{-1}(B) \in \Sigma\}$$

is called **the push-forward of Σ by f on** Y .

(i) Prove that the collection $\{B \subseteq Y \mid f^{-1}(B) \in \Sigma\}$ is a σ-algebra of subsets of Y .

Consider also a σ-algebra Σ^0 of subsets of Y and a collection E of subsets of Y so that $\Sigma(E) = \Sigma^0$.

(ii) Prove that, if $f^{-1}(B) \in \Sigma$ for all $B \in$ E, then $f^{-1}(B) \in \Sigma$ for all $B \in \Sigma^0$.

(iii) If X,Y are two topological spaces and $f : X \to Y$ is continuous, prove that $f^{-1}(B)$ is a Borel set in X for every Borel set B in Y .

3.The pull-back of a σ-algebra.

Let Σ^0 be a σ-algebra of subsets of Y and let $f : X \to Y$. Then the collection

$$\{f^{-1}(B) \mid B \in \Sigma^0\}$$

is called **the pull-back of Σ^0 by f on** X.

Prove that $\{f^{-1}(B) \mid B \in \Sigma^0\}$ is a σ-algebra of subsets of X.

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1.11 ANSWERS TO CHECK YOUR PROGRESS

1.please check section 1.3-1.7 for answers
2.please check section 1.3-1.7 for answers
3.please check section 1.3-1.7 for answers
4.please check section 1.3-1.7 for answers

UNIT 2 MEASURES

STRUCTURE

2.1 OBJECTIVES

In this Unit we are going to study about general measures, Point mass distributions, complete measures, restrictions and its uniqueness.

2.2 INTRODUCTION

Let (X,Σ) *be a measurable space. A function* $\mu : \Sigma \to [0,+\infty]$ *is called a measure on* (X,Σ).

The triple (X,Σ,μ) *of a non-empty set X, a σ-algebra of subsets of X and a measure* μ *on* Σ *is called a measure space.*

2.3 GENERAL MEASURES.

Definition 2.1 *Let* (X,Σ) *be a measurable space. A function* $\mu : \Sigma \to [0,+\infty]$ *is called a measure on* (X,Σ) *if* *(i)* $\mu(\emptyset) = 0$, *(ii)* $\mu(\cup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} \mu(A_n)$ *for all sequences* (A_n) *of pairwise disjoint sets which are contained in* Σ.

The triple (X,Σ,μ) *of a non-empty set* X*, a* σ*-algebra of subsets of* X *and a measure* μ *on* Σ *is called a measure space.*

For simplicity and if there is no danger of confusion, we shall say that μ is **a measure on** Σ or **a measure on** X.

Note that the values of a measure are non-negative real numbers or $+\infty$.

Property (ii) of a measure is called σ**-additivity** and sometimes a measure is also called σ**-additive measure** to distinguish from a so-called **finitely additive measure** μ which is defined to satisfy $\mu(\emptyset) = 0$ and $\mu(\cup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \mu(A_n)$ for all $N \in \mathbf{N}$ and all pairwise disjoint $A_1,\dots,A_N \in \Sigma$.

Proposition 2.1 *Every measure is finitely additive.*

Proof: Let μ be a measure on the σ-algebra Σ. If $A_1,\dots,A_N \in \Sigma$ are pairwise disjoint, we consider $A_n = \emptyset$ for all $n > N$ and we get $\mu(\cup_{n=1}^{N} A_n) = \mu(\cup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} \mu(A_n) = \sum_{n=1}^{N} \mu(A_n)$.

Examples.

1. The simplest measure is the *zero measure* which is denoted o and is defined by $o(A) = 0$ for every $A \in \Sigma$.

2. Let X be an uncountable set and consider $\Sigma = \{A \subseteq X \mid A$ is countable or A^c is countable$\}$. We define $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A^c is countable.

Then it is clear that $\mu(\emptyset) = 0$ and let $A_1,A_2,\dots \in \Sigma$ be pairwise disjoint. If all of them are countable, then $\cup_{n=1}^{+\infty} A_n$ is also countable and we get $\mu(\cup_{n=1}^{+\infty} A_n) = 0 = \sum_{n=1}^{+\infty} \mu(A_n)$. Observe that if one of the A_n's, say A_{n0}, is uncountable, then for all $n \neq n_0$ we have $A_n \subseteq A_{n_0}^c$ which is countable. Therefore $\mu(A_{n0}) = 1$ and $\mu(A_n) = 0$ for all $n \neq n_0$. Since $(\cup_{n=1}^{+\infty} A_n)^c (\subseteq A_{n_0}^c)$ is countable, we get $\mu(\cup_{n=1}^{+\infty} A_n) = 1 = \sum_{n=1}^{+\infty} \mu(A_n)$.

Theorem 2.1 *Let* (X,Σ,μ) *be a measure space.*

(i) *(Monotonicity) If* $A,B \in \Sigma$ *and* $A \subseteq B$*, then* $\mu(A) \leq \mu(B)$.

(ii) *If* $A,B \in \Sigma$, $A \subseteq B$ *and* $\mu(A) < +\infty$*, then* $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(iii) *(*σ*-subadditivity) If* $A_1,A_2,\dots \in \Sigma$*, then* $\mu(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu(A_n)$.

(iv) (Continuity from below) If $A_1,A_2,... \in \Sigma$ *and* $A_n \uparrow A$*, then* $\mu(A_n) \uparrow \mu(A)$*. (v) (Continuity from above) If* $A_1,A_2,... \in \Sigma$*,* $\mu(A_N) < +\infty$ *for some N and* $A_n \downarrow A$*, then* $\mu(A_n) \downarrow \mu(A)$*.*

Proof: (i) We write $B = A \cup (B \setminus A)$. By finite additivity of μ, $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$.

(ii) From both sides of $\mu(B) = \mu(A) + \mu(B \setminus A)$ we subtract $\mu(A)$.

(iii) Using Proposition 1.2 we find $B_1,B_2,... \in \Sigma$ which are pairwise disjoint and satisfy $B_n \subseteq A_n$ for all n and $\cup_{n=1}^{+\infty} B_n = \cup_{n=1}^{+\infty} A_n$. By σ-additivity and monotonicity of μ we get $\mu(\cup_{n=1}^{+\infty} A_n) = \mu(\cup_{n=1}^{+\infty} B_n) = \sum_{n=1}^{+\infty} \mu(B_n) \leq \sum_{n=1}^{+\infty} \mu(A_n)$.

(iv) We write $A = A_1 \cup \cup_{k=1}^{+\infty}(A_{k+1} \setminus A_k)$), where all sets whose union is taken in the right side are pairwise disjoint. Applying σ-additivity (and finite additivity),

$$\mu(A) = \mu(A_1) + \sum_{k=1}^{+} \mu(A_{k+1} \setminus A_k) = \lim_{n\to+\infty}[\mu(A_1) + \sum_{k=1}^{-1} \mu(A_{k+1} \setminus A_k)] = \lim_{n\to+\infty} \mu\big(A_1 \cup \cup_{k=1}^{n-1}(A_{k+1} \setminus A_k)\big) = \lim_{n\to+\infty} \mu(A_n).$$

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(v) We observe that $A_N \setminus A_n \uparrow A_N \setminus A$ and continuity from below implies $\mu(A_N \setminus A_n) \uparrow \mu(A_N \setminus A)$. Now, $\mu(A_N) < +\infty$ implies $\mu(A_n) < +\infty$ for all $n \geq N$ and $\mu(A) < +\infty$. Applying (ii), we get $\mu(A_N) - \mu(A_n) \uparrow \mu(A_N) - \mu(A)$ and, since $\mu(A_N) < +\infty$, we find $\mu(A_n) \downarrow \mu(A)$.

Definition 2.2 *Let* (X,Σ,μ) *be a measure space.*

(i) μ *is called finite if* $\mu(X) < +\infty$*.*

(ii) μ *is called σ-finite if there exist* $X_1,X_2,... \in \Sigma$ *so that* $X = \cup_{n=1}^{+\infty} X_n$ *and* $\mu(X_n) < +\infty$ *for all* $n \in \mathbf{N}$.

(iii) μ *is called semifinite if for every* $E \in \Sigma$ *with* $\mu(E) = +\infty$ *there is an* $F \in \Sigma$ *so that* $F \subseteq E$ *and* $0 < \mu(F) < +\infty$.

(iv) A set $E \in \Sigma$ *is called of finite μ-measure if* $\mu(E) < +\infty$*.*

(v) A set $E \in \Sigma$ *is called of σ-finite μ-measure if there exist* $E_1,E_2,... \in \Sigma$ *so that* $E \subseteq \cup_{n=1}^{+\infty} E_n$ *and* $\mu(E_n) < +\infty$ *for all n.*

For simplicity and if there is no danger of confusion, we may say that E is *of finite measure* or *of σ-finite measure*.

Some observations related to the last definition are immediate.

1. If μ is finite then all sets in Σ are of finite measure. More generally, if $E \in \Sigma$ is of finite measure, then all subsets of it in Σ are of finite measure.

2. If μ is σ-finite then all sets in Σ are of σ-finite measure. More generally, if $E \in \Sigma$ is of σ-finite measure, then all subsets of it in Σ are of σ-finite measure.

3. The collection of sets of finite measure is closed under finite unions.

4. The collection of sets of σ-finite measure is closed under countable unions.

5. If μ is σ-finite, applying Proposition 1.2, we see that there exist *pairwise disjoint* $X_1, X_2, \ldots \in \Sigma$ so that $X = \cup_{n=1}^{+\infty} X_n$ and $\mu(X_n) < +\infty$ for all n.

Similarly, by taking successive unions, we see that there exist $X_1, X_2, \ldots \in \Sigma$ so that $X_n \uparrow X$ and $\mu(X_n) < +\infty$ for all n. We shall use these two observations freely whenever σ-finiteness appears in the sequel.

6. If μ is finite, then it is also σ-finite. The next result is not so obvious.

Proposition 2.2 *Let (X,Σ,μ) be a measure space. If μ is σ-finite, then it is semifinite.*

Proof: Take $X_1, X_2, \ldots \in \Sigma$ so that $X_n \uparrow X$ and $\mu(X_n) < +\infty$ for all n. Let $E \in \Sigma$ have $\mu(E) = +\infty$. From $E \cap X_n \uparrow E$ and continuity of μ from below, we get $\mu(E \cap X_n) \uparrow +\infty$. Therefore, $\mu(E \cap X_{n0}) > 0$ for some n_0 and we observe that $\mu(E \cap X_{n0}) \leq \mu(X_{n0}) < +\infty$.

Definition 2.3 *Let (X,Σ,μ) be a measure space. $E \in \Sigma$ is called μ-null if* $\mu(E) = 0$.

For simplicity and if there is no danger of confusion, we may say that E is *null* instead of μ-null.

The following is trivial but basic.

Theorem 2.2 *Let (X,Σ,μ) be a measure space.*

(i) If $E \in \Sigma$ is null, then every subset of it in Σ is also null.

(ii) If $E_1, E_2, \ldots \in \Sigma$ are all null, then their union $\cup_{n=1}^{+\infty} E_n$ is also null.

Proof: The proof is based on the monotonicity and the σ-subadditivity of μ.

Check your progress

1.Let (X,Σ,μ) be a measure space and $Y \in \Sigma$ be non-empty. Prove that μ_Y is the only measure on (X,Σ) with the properties: (i) $\mu_Y(E) = \mu(E)$ for all $E \in \Sigma$ with $E \subseteq Y$, (ii) $\mu_Y(E) = 0$ for all $E \in \Sigma$ with $E \subseteq Y^c$.

2.4 POINT MASS DISTRIBUTION

Before introducing a particular class of measures we shall define *sums of nonnegative terms over general sets of indices*. We shall follow the standard practice of using both notations $a(i)$ and a_i for the values of a function a on a set I of indices.

Definition 2.4 *Let I be a non-empty set of indices and* $a : I \to [0,+\infty]$. *We define the sum of the values of a by*

$$\sum_{i\in I} a_i = \sup\left\{ \sum_{i\in F} a_i \,\middle|\, F \text{ non-empty finite subset of } I \right\}.$$

If $I = \emptyset$, *we define* $\sum_{i\in I} a_i = 0$.

Of course, if F is a non-empty finite set, then $\sum_{i\in F} a_i$ is just equal to the sum

$\sum_{k=1}^{N} a_{i_k}$, where $F = \{a_{i1},...,a_{iN}\}$ is an arbitrary enumeration of F.

We first make sure that this definition extends a simpler situation.

Proposition 2.3 *If I is countable and* $I = \{i_1,i_2,...\}$ *is an arbitrary enumeration of it, then* $\sum_{i\in I} a_i = \sum_{k=1}^{+\infty} a_{i_k}$ *for all* $a : I \to [0,+\infty]$.

Proof: For arbitrary N we consider the finite subset $F = \{i_1,...,i_N\}$ of I. Then, by the definition of $\sum_{i\in I} a_i$, we have $\sum_{k=1}^{N} a_{i_k} = \sum_{i\in F} a_i \le \sum_{i\in I} a_i$. Since N is arbitrary, we find $\sum_{k=1}^{+\infty} a_{i_k} \le \sum_{i\in I} a_i$.

Now for an arbitrary non-empty finite $F \subseteq I$ we consider the indices of the elements of F provided by the enumeration $I = \{i_1,i_2,...\}$ and take the maximal, say N, of them. This means that $F \subseteq \{i_1,i_2,...,i_N\}$. Therefore $\sum_{i\in F} a_i \le \sum_{k=1}^{N} a_{i_k} \le \sum_{k=1}^{+\infty} a_{i_k}$ and, since F is arbitrary, we find, by the definition of $\sum_{i\in I} a_i$, that $\sum_{i\in I} a_i \le \sum_{k=1}^{+\infty} a_{i_k}$.

Proposition 2.4 *Let* $a : I \to [0,+\infty]$. *If* $\sum_{i\in I} a_i < +\infty$, *then* $a_i < +\infty$ *for all i and the set* $\{i \in I \mid a_i > 0\}$ *is countable.*

Proof: Let $\sum_{i\in I} a_i < +\infty$. It is clear that $a_i < +\infty$ for all i (take $F = \{i\}$) and, for arbitrary n, consider the set $I_n = \{i \in I \,|\, a_i \geq \frac{1}{n}\}$. If F is an arbitrary finite subset of I_n, then $\frac{1}{n}\text{card}(F) \leq \sum_{i\in F} a_i \leq \sum_{i\in I} a_i$. Therefore, the cardinality of the arbitrary finite subset of I_n is not larger than the number $n \sum_{i\in I} a_i$ and, hence, the set I_n is finite. But then, $\{i \in I \,|\, a_i > 0\} = \cup_{n=1}^{+\infty} I_n$ is countable.

Proposition 2.5 *(i) If $a,b : I \to [0,+\infty]$ and $a_i \leq b_i$ for all $i \in I$, then $\sum_{i\in I} a_i \leq \sum_{i\in I} b_i$.*

(ii) If $a : I \to [0,+\infty]$ and $J \subseteq I$, then $\sum_{i\in J} a_i \leq \sum_{i\in I} a_i$.

Proof: (i) For arbitrary finite $F \subseteq I$ we have $\sum_{i\in F} a_i \leq \sum_{i\in F} b_i \leq \sum_{i\in I} b_i$. Taking supremum over the finite subsets of I, we find $\sum_{i\in I} a_i \leq \sum_{i\in I} b_i$.

(ii) For arbitrary finite $F \subseteq J$ we have that $F \subseteq I$ and hence $\sum_{i\in F} a_i \leq \sum_{i\in I} a_i$.

Taking supremum over the finite subsets of J, we get $\sum_{i\in J} a_i \leq \sum_{i\in I} a_i$.

Proposition 2.6 *Let $I = \cup_{k\in K} J_k$, where K is a non-empty set of indices and the J_k's are non-empty and pairwise disjoint. Then for every $a : I \to [0,+\infty]$ we have* $\sum_{i\in I} a_i = \sum_{k\in K} \left(\sum_{i\in J_k} a_i\right)$.

Proof: Take an arbitrary finite $F \subseteq I$ and consider the finite sets $F_k = F \cap J_k$. Observe that the set $L = \{k \in K \,|\, F_k \neq \emptyset\}$ is a finite subset of K. Then, using trivial properties of sums over finite sets of indices, we find $\sum_{i\in F} a_i = \sum_{k\in L} \left(\sum_{i\in F_k} a_i\right)$. The definitions imply that $\sum_{i\in F} a_i \leq \sum_{k\in L} \left(\sum_{i\in J_k} a_i\right) \leq \sum_{k\in K} \left(\sum_{i\in J_k} a_i\right)$. Taking supremum over the finite subsets F of I we find $\sum_{i\in I} a_i \leq \sum_{k\in K} \sum_{i\in J_k} a_i$.

Now take an arbitrary finite $L \subseteq K$ and arbitrary finite $F_k \subseteq J_k$ for each $k \in L$. Then $\sum_{k\in L} \left(\sum_{i\in F_k} a_i\right)$ is, clearly, a sum (without repetitions) over the finite subset $\cup_{k\in L} F_k$ of I. Hence $\sum_{k\in L} \left(\sum_{i\in F_k} a_i\right) \leq \sum_{i\in I} a_i$. Taking supremum over the finite subsets F_k of J_k for each $k \in L$, one at a time, we get that $\sum_{k\in L} \left(\sum_{i\in J_k} a_i\right) \leq \sum_{i\in I} a_i$. Finally, taking supremum over the finite subsets L of K, we find $\sum_{k\in K} \left(\sum_{i\in J_k} a_i\right) \leq \sum_{i\in I} a_i$ and conclude the proof.

After this short investigation of the general summation notion we define a class of measures.

Proposition 2.7 *Let X be non-empty and consider $a : X \to [0,+\infty]$. We define $\mu : \mathcal{P}(X) \to [0,+\infty]$ by*

$$\mu(E) = \sum_{x\in E} a_x, \qquad E \subseteq X.$$

Then μ is a measure on $(X,\mathcal{P}(X))$.

Proof: It is obvious that $\mu(\emptyset) = \sum_{x\in\emptyset} a_x = 0$.

If $E_1,E_2,\ldots$ are pairwise disjoint and $E = \cup_{n=1}^{+\infty} E_n$, we apply Propositions 2.3 and 2.6 to find $\mu(E) = \sum_{x\in E} a_x = \sum_{n\in\mathbf{N}} \left(\sum_{x\in E_n} a_x\right) = \sum_{n\in\mathbf{N}} \mu(E_n) = \sum_{n=1}^{+\infty} \mu(E_n)$.

Definition 2.5 *The measure on $(X,\mathcal{P}(X))$ defined in the statement of the previous proposition is called the point-mass distribution on X induced by the function a. The value a_x is called the point-mass at x.*

Examples.

Consider the function which puts point-mass $a_x = 1$ at every $x \in X$. It is then obvious that the induced point-mass distribution is

$$]\,(E) = \begin{cases} \mathrm{card}(E), & \text{if } E \text{ is a finite} \subseteq X, \\ +\infty, & \text{if } E \text{ is an infinite} \subseteq X. \end{cases}$$

This $]$ is called **the counting measure on** X.

Take a particular $x_0 \in X$ and the function which puts point-mass $a_{x_0} = 1$ at x_0 and point-mass $a_x = 0$ at all other points of X. Then the induced point-mass distribution is

$$\delta_{x_0}(E) = \begin{cases} 1, & \text{if } x_0 \in E \subseteq X \\ 0, & \text{if } x_0 \notin E \subseteq X \end{cases}$$

This δ_{x_0} is called **the Dirac measure at** x_0 or **the Dirac mass at** x_0.

Of course, it is very easy to show directly, without using Proposition 2.7, that these two examples, $]$ and δ_{x_0}, constitute measures.

Check your progress

2. Positive linear combinations of measures.

Let $\mu,\mu_1\mu_2$ be measures on the measurable space (X,Σ) and $\kappa \in [0,+\infty)$.

(i) Prove that $\kappa\mu : \Sigma \to [0,+\infty]$, which is defined by

$$(\kappa\mu)(E) = \kappa \cdot \mu(E), \qquad E \in \Sigma,$$

(consider $0 \cdot (+\infty) = 0$) is a measure on (X,Σ). This $\kappa\mu$ is called **the product of μ by κ.**

(ii) Prove that $\mu_1 + \mu_2 : \Sigma \to [0,+\infty]$, which is defined by $(\mu_1 + \mu_2)(E) = \mu_1(E) + \mu_2(E)$, $E \in \Sigma$, is a measure on (X,Σ). This $\mu_1 + \mu_2$ is called **the sum of μ_1 and μ_2.**

Thus, we define **positive linear combinations** $\kappa_1\mu_1 + \cdots + \kappa_n\mu_n$.

2.5 COMPLETE MEASURES.

Theorem 2.2(i) says that a subset of a μ-null set is also μ-null, *provided* that the subset is contained in the σ-algebra on which the measure μ is defined.

Definition 2.6 *Let (X,Σ,μ) be a measure space. Suppose that for every $E \in \Sigma$ with $\mu(E) = 0$ and every $F \subseteq E$ it is implied that $F \in \Sigma$ (and hence $\mu(F) = 0$, also). Then μ is called complete and (X,Σ,μ) is a complete measure space.*

Thus, a measure μ is complete if the σ-algebra on which it is defined contains all subsets of μ-null sets.

Definition 2.7 *If (X,Σ_1,μ_1) and (X,Σ_2,μ_2) are two measure spaces on the same set X, we say that (X,Σ_2,μ_2) is an extension of (X,Σ_1,μ_1) if $\Sigma_1 \subseteq \Sigma_2$ and $\mu_1(E) = \mu_2(E)$ for all $E \in \Sigma_1$.*

Theorem 2.3 *Let (X,Σ,μ) be a measure space. Then there is a unique smallest complete extension (X,Σ,μ) of (X,Σ,μ). Namely, there is a unique measure space (X,Σ,μ) so that*

(i) $(\overline{X},\overline{\Sigma},\mu)$ *is an extension of* (X,Σ,μ),

(ii) $(X,\overline{\Sigma},\mu)$ *is complete,*

(iii) if $(\overline{\overline{X}},\overline{\overline{\Sigma}},\mu)$ *is another complete extension of* (X,Σ,μ)*, then it is an extension also of* (X,Σ,μ)*.*

Proof: We shall first construct $(\overline{X},\overline{\Sigma},\mu)$. We define

$$\overline{\Sigma} = \{A \cup F \mid A \in \Sigma \text{ and } F \subseteq E \text{ for some } E \in \Sigma \text{ with } \mu(E) = 0\}.$$

We prove that $\overline{\Sigma}$ is a σ-algebra. We write $\emptyset = \emptyset \cup \emptyset$, where the first $\emptyset$ belongs to Σ and the second $\emptyset$ is a subset of $\emptyset \in \Sigma$ with $\mu(\emptyset) = 0$. Therefore $\emptyset \in \overline{\Sigma}$.

Let $B \in \overline{\Sigma}$. Then $B = A \cup F$ for $A \in \Sigma$ and $F \subseteq$ of some $E \in \Sigma$ with $\mu(E) = 0$. Write $B^c = A_1 \cup F_1$, where $A_1 = (A \cup E)^c$ and $F_1 = E \setminus (A \cup F)$. Then $A_1 \in \Sigma$ and $F_1 \subseteq E$. Hence $B^c \in \overline{\Sigma}$.

Let $B_1, B_2, \ldots \in \overline{\Sigma}$. Then for every n, $B_n = A_n \cup F_n$ for $A_n \in \Sigma$ and $F_n \subseteq$ of some $E_n \in \Sigma$ with $\mu(E_n) = 0$. Now $\cup_{n=1}^{+\infty} B_n = (\cup_{n=1}^{+\infty} A_n) \cup (\cup_{n=1}^{+\infty} F_n)$, where $\cup_{n=1}^{+\infty} A_n \in \Sigma$ and $\cup_{n=1}^{+\infty} F_n \subseteq \cup_{n=1}^{+\infty} E_n \in \Sigma$ with $\mu(\cup_{n=1}^{+\infty} E_n) = 0$. Therefore $\cup_{n=1}^{+\infty} B_n \in \overline{\Sigma}$.

We now construct $\overline{\mu}$. For every $B \in \overline{\Sigma}$ we write $B = A \cup F$ for $A \in \Sigma$ and $F \subseteq$ of some $E \in \Sigma$ with $\mu(E) = 0$ and define

$$\overline{\mu}(B) = \mu(A).$$

To prove that $\overline{\mu}(B)$ is well defined we consider that we may also have $B = A^0 \cup F^0$ for $A^0 \in \Sigma$ and $F^0 \subseteq$ of some $E^0 \in \Sigma$ with $\mu(E^0) = 0$ and we must prove that $\mu(A) = \mu(A^0)$. Since $A \subseteq B \subseteq A^0 \cup E^0$, we have $\mu(A) \leq \mu(A^0) + \mu(E^0) = \mu(A^0)$ and, symmetrically, $\mu(A^0) \leq \mu(A)$.

To prove that $\overline{\mu}$ is a measure on $(X, \overline{\Sigma})$ let $\emptyset = \emptyset \cup \emptyset$ as above and get $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$. Let also $B_1, B_2, \ldots \in \overline{\Sigma}$ be pairwise disjoint. Then $B_n = A_n \cup F_n$ for $A_n \in \Sigma$ and $F_n \subseteq E_n \in \Sigma$ with $\mu(E_n) = 0$. Observe that the A_n's are pairwise disjoint. Then $\cup_{n=1}^{+\infty} B_n = (\cup_{n=1}^{+\infty} A_n) \cup (\cup_{n=1}^{+\infty} F_n)$ and $\cup_{n=1}^{+\infty} F_n \subseteq \cup_{n=1}^{+\infty} E_n \in \Sigma$ with $\mu(\cup_{n=1}^{+\infty} E_n) = 0$. Therefore $\overline{\mu}(\cup_{n=1}^{+\infty} B_n) = \mu(\cup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} \mu(A_n) = \sum_{n=1}^{+\infty} \overline{\mu}(B_n)$.

We now prove that $\overline{\mu}$ is complete. Let $B \in \overline{\Sigma}$ with $\overline{\mu}(B) = 0$ and let $B^0 \subseteq B$. Write $B = A \cup F$ for $A \in \Sigma$ and $F \subseteq E \in \Sigma$ with $\mu(E) = 0$ and observe that $\mu(A) = \overline{\mu}(B) = 0$. Then write $B^0 = A^0 \cup F^0$, where $A^0 = \emptyset \in \Sigma$

and $B' \in \overline{\Sigma}$. $F' = B' \subseteq E'$ where $E^0 = A \cup E \in \Sigma$ with $\mu(E^0) \leq \mu(A) + \mu(E) = 0$. Hence

To prove that $(X,\overline{\Sigma},\overline{\mu})$ is an extension of (X,Σ,μ) we take any $A \in \Sigma$ and write $A = A \cup \emptyset$, where $\emptyset \subseteq \emptyset \in \Sigma$ with $\mu(\emptyset) = 0$. This implies that $A \in \overline{\Sigma}$ and $\overline{\mu}(A) = \mu(A)$.

Now suppose that $(X,\overline{\overline{\Sigma}},\overline{\overline{\mu}})$ is another complete extension of (X,Σ,μ). Take any $B \in \overline{\Sigma}$ and thus $B = A \cup F$ for $A \in \Sigma$ and $F \subseteq E \in \Sigma$ with $\mu(E) = 0$.

But then $A,E \in \overline{\overline{\Sigma}}$ and $\overline{\overline{\mu}}(E) = \mu(E) = 0$. Since $\overline{\overline{\mu}}$ is complete, we get that also

$F \in \overline{\overline{\Sigma}}$ and hence $B = A \cup F \in \overline{\overline{\Sigma}}$.

Moreover, $\overline{\overline{\mu}}(A) \leq \overline{\overline{\mu}}(B) \leq \overline{\overline{\mu}}(A) + \overline{\overline{\mu}}(F) = \overline{\overline{\mu}}(A)$, which implies $\overline{\overline{\mu}}(B) = \overline{\overline{\mu}}(A) = \mu(A) = \overline{\mu}(B)$.

It only remains to prove the uniqueness of a smallest complete extension of (X,Σ,μ). This is obvious, since two *smallest* complete extensions of (X,Σ,μ) must both be extensions of each other and, hence, identical.

Definition 2.8 *If* (X,Σ,μ) *is a measure space, then its smallest complete extension is called the completion of* (X,Σ,μ).

2.6 RESTRICTION OF A MEASURE.

Proposition 2.8 *Let* (X,Σ,μ) *be a measure space and let* $Y \in \Sigma$. *If we define* $\mu_Y : \Sigma \to [0,+\infty]$ *by*

$$\mu_Y(A) = \mu(A \cap Y), \qquad A \in \Sigma,$$

then μ_Y *is a measure on* (X,Σ) *with the properties that* $\mu_Y(A) = \mu(A)$ *for every* $A \in \Sigma$, $A \subseteq Y$, *and that* $\mu_Y(A) = 0$ *for every* $A \in \Sigma$, $A \cap Y = \emptyset$.

Proof: We have $\mu_Y(\emptyset) = \mu(\emptyset \cap Y) = \mu(\emptyset) = 0$. If $A_1,A_2,... \in \Sigma$ are pairwise disjoint, $\mu_Y(\cup_{j=1}^{+\infty} A_j) = \mu\big((\cup_{j=1}^{+\infty} A_j) \cap Y\big) =$
$\mu\big(\cup_{j=1}^{+\infty}(A_j \cap Y)\big) = \sum_{j=1}^{+\infty} \mu(A_j \cap Y) = \sum_{j=1}^{+\infty} \mu_Y(A_j)$.

Therefore, μ_Y is a measure on (X,Σ) and its two properties are trivial to prove.

Definition 2.9 *Let* (X,Σ,μ) *be a measure space and let* $Y \in \Sigma$. *The measure* μ_Y *on* (X,Σ) *of Proposition 2.8 is called the* Y *-restriction of* μ.

There is a second kind of restriction of a measure. To define it we recall that, if $Y \in \Sigma$, the restriction $\Sigma \mathrm{e} Y$ of the σ-algebra Σ of subsets of X on the non-empty $Y \subseteq X$ is $\Sigma \mathrm{e} Y = \{A \subseteq Y \mid A \in \Sigma\}$.

Proposition 2.9 *Let* (X,Σ,μ) *be a measure space and let* $Y \in \Sigma$ *be non-empty. We consider* $\Sigma \mathrm{e} Y = \{A \subseteq Y \mid A \in \Sigma\}$ *and define* $\mu \mathrm{e} Y : \Sigma \mathrm{e} Y \to [0,+\infty]$ *by*

$$(\mu \mathrm{e} Y)(A) = \mu(A), \qquad A \in \Sigma \mathrm{e} Y.$$

Then $\mu \mathrm{e} Y$ *is a measure on* $(Y,\Sigma \mathrm{e} Y)$.

Proof: Obvious.

Definition 2.10 *Let* (X,Σ,μ) *be a measure space and let* $Y \in \Sigma$ *be non-empty. The measure* $\mu \mathrm{e} Y$ *on* $(Y,\Sigma \mathrm{e} Y)$ *of Proposition 2.9 is called the restriction of* μ *on* $\Sigma \mathrm{e} Y$.

Informally speaking, we may describe the relation between the two restrictions of μ as follows. The restriction μ_Y assigns value 0 to all sets in Σ which are included in the complement of Y while the restriction $\mu \mathrm{e} Y$ simply ignores all those sets. Both restrictions μ_Y and $\mu \mathrm{e} Y$ assign the same values (the same to the values that μ assigns) to all sets in Σ which are included in Y.

Check your progress

3 Let X be non-empty and consider a finite $A \subseteq X$. If $a : X \to [0,+\infty]$ satisfies $a_x = 0$ for all $x \notin A$, prove that the point-mass distribution μ on X induced by a can be written as a positive linear combination of Dirac measures:

$$\mu = \kappa_1\delta_{x1} + \cdots + \kappa_k\delta_{xk}.$$

Let X be infinite and define for all $E \subseteq X$

$$\mu(E) = \begin{cases} 0, & \text{if } E \text{ is finite,} \\ +\infty, & \text{if } E \text{ is infinite.} \end{cases}$$

Prove that μ is a finitely additive measure on $(X,\mathcal{P}(X))$ which is not a measure.

2.7 UNIQUENESS OF MEASURES.

The next result is very useful when we want to prove that two measures are equal on a σ-algebra Σ. It says that it is enough to prove that they are equal on an algebra which generates Σ, provided that an extra assumption of σ-finiteness of the two measures on the algebra is satisfied.

Theorem 2.4 *Let* A *be an algebra of subsets of X and let μ, ν be two measures on* $(X,\Sigma(\mathrm{A}))$. *Suppose there exist $A_1, A_2, \ldots \in$ A with $A_n \uparrow X$ and $\mu(A_k), \nu(A_k) < +\infty$ for all k.*

If μ, ν are equal on A, *then they are equal also on* $\Sigma(\mathrm{A})$.

Proof: (a) Suppose that $\mu(X), \nu(X) < +\infty$.

We define the collection $\mathrm{M} = \{E \in \Sigma(\mathrm{A}) | \mu(E) = \nu(E)\}$. It is easy to see that M is a monotone class. Indeed, let $E_1, E_2, \ldots \in \mathrm{M}$ with $E_n \uparrow E$. By continuity of measures from below, we get $\mu(E) = \lim_{n\to+\infty} \mu(E_n) = \lim_{n\to+\infty} \nu(E_n) = \nu(E)$ and thus $E \in \mathrm{M}$. We do exactly the same when $E_n \downarrow E$, using the continuity of measures from above and the extra assumption $\mu(X), \nu(X) < +\infty$.

Since M is a monotone class including A, Proposition 1.7 implies that $\mathrm{M}(\mathrm{A}) \subseteq \mathrm{M}$. Now, Theorem 1.1 implies that $\Sigma(\mathrm{A}) \subseteq \mathrm{M}$ and, thus, $\mu(E) = \nu(E)$ for all $E \in \Sigma(\mathrm{A})$.

(b) The general case.

For each k, we consider the A_k-restrictions of μ, ν. Namely,

$$\mu_{A_k}(E) = \mu(E \cap A_k), \qquad \nu_{A_k}(E) = \nu(E \cap A_k)$$

for all $E \in \Sigma(\mathrm{A})$. All μ_{A_k} and ν_{A_k} are finite measures on (X,Σ), because $\mu_{A_k}(X) = \mu(A_k) < +\infty$ and $\nu_{A_k}(X) = \nu(A_k) < +\infty$. We, clearly, have that μ_{A_k}, ν_{A_k} are equal on A and, by the result of (a), they are equal also on $\Sigma(\mathrm{A})$.

For every $E \in \Sigma(\mathrm{A})$, using the $E \cap A_k \uparrow E$ and the continuity of measures from below, we can write $\mu(E) = \lim_{k\to+\infty} \mu(E \cap A_k) = \lim_{k\to+\infty} \mu_{A_k}(E) = \lim_{k\to+\infty} \nu_{A_k}(E) = \lim_{k\to+\infty} \nu(E \cap A_k) = \nu(E)$.

Thus, μ, ν are equal on $\Sigma(\mathrm{A})$.

2.8 LET US SUM UP

In this unit we discussed the following

- General measure
- Uniqueness of measures
- Restriction of measure
- Point mass distribution
- Complete measures

2.9 KEYWORDS

Meausre- In mathematics, a measure is a generalisation of the concepts as length, area and volume. Informally, measures may be regarded as "mass distributions". More precisely, a measure is a function that assigns a number to certain subsets of a given set. This number is said to be the measure of the set.

Distribution-Algebraic **distribution** means to multiply each of the terms within the parentheses by another term that is outside the parentheses.

2.10 QUESTIONS FOR REVIEW

1.Let (X,Σ,μ) be a measure space and $E \in \Sigma$ be of σ-finite measure. If $\{D_i\}_{i\in I}$ is a collection of pairwise disjoint sets in Σ, prove that the set $\{i \in I \mid \mu(E \cap D_i) > 0\}$ is countable.

3. Let X be uncountable and define for all $E \subseteq X$

$$\mu(E) = \begin{cases} 0, & \text{if } E \text{ is countable,} \\ +\infty, & \text{if } E \text{ is uncountable.} \end{cases}$$

Prove that μ is a measure on $(X,\mathrm{P}(X))$ which is not semifinite.

2.Let (X,Σ,μ) be a complete measure space. If $A \in \Sigma$, $B \subseteq X$ and $\mu(A4B) = 0$, prove that $B \in \Sigma$ and $\mu(B) = \mu(A)$.

3.Let μ be a finitely additive measure on the measurable space (X,Σ).

(i) Prove that μ is a measure if and only if it is continuous from below. (ii) If $\mu(X) < +\infty$, prove that μ is a measure if and only if it is continuous from above.

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2.12 ANSWER TO CHECK YOUR PROGRESS

1. Please check the session 2.3 (Answer to QUESTION 1)
2. Please check the session 2.4(Answer to QUESTION 2)
3. Please check the session 2.6(Answer to QUESTION 2)

UNIT 3 OUTER MEASURES

STRUCTURE

3.1 OBJECTIVES

In this chapter we are going to learn about outer measures and its constructions and solve problems related to it.

3.2 INTRODUCTION

An outer measure on X is defined for *all* subsets of X, it is monotone and σ-subadditive. An outer measure is also finitely subadditive, because for every $A_1,\ldots,A_N \subseteq X$ we set $A_n = \emptyset$ for all $n > N$ and get $\mu^*(\cup_{n=1}^{N} A_n) =$

$$\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) = \sum_{n=1} \mu^*(A_n).N$$

3.3 OUTER MEASURES.

Definition 3.1 *Let X be a non-empty set. A function* $\mu^* : \mathrm{P}(X) \to [0,+\infty]$ *is called outer measure on X if (i)* $\mu^*(\emptyset) = 0$,

(ii) $\mu^*(A) \leq \mu^*(B)$ *if* $A \subseteq B \subseteq X$, *(iii)* $\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu^*(A_n)$ *for all sequences* (A_n) *of subsets of X.*

Thus, an outer measure on X is defined for *all* subsets of X, it is monotone and σ-subadditive. An outer measure is also finitely subadditive, because for every $A_1,\ldots,A_N \subseteq X$ we set $A_n = \emptyset$ for all $n > N$ and get $\mu^*(\cup_{n=1}^{N} A_n) =$

$$\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) = \sum_{n=1} \mu^*(A_n).N$$

We shall see now how a measure is constructed from an outer measure.

Definition 3.2 *Let μ^* be an outer measure on the non-empty set X. We say that the set $A \subseteq X$ is μ^*-measurable if*

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E)$$

for all $E \subseteq X$.

We denote Σ_{μ^*} the collection of all μ^*-measurable subsets of X.

Thus, a set is μ^*-measurable if and only if it decomposes every subset of X into two disjoint pieces whose outer measures add to give the outer measure of the subset.

Observe that $E = (E \cap A) \cup (E \cap A^c)$ and, by the subadditivity of μ^*, we have $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Therefore, in order to check the validity of the equality in the definition, it is enough to check the inequality

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E).$$

Furthermore, it is enough to check this last inequality in the case $\mu^*(E) < +\infty$.

3.4 CONSTRUCTION OF OUTER MEASURES.

Theorem 3.2 *Let* C *be a collection of subsets of X, containing at least the* $\emptyset$*, and $\tau : \mathcal{C} \to [0,+\infty]$ be an arbitrary function with $\tau(\emptyset) = 0$. We define*

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{+\infty} \tau(C_j) \,\middle|\, C_1, C_2, \ldots \in \mathcal{C} \text{ so that } E \subseteq \cup_{j=1}^{+\infty} C_n \right\}$$

for all $E \subseteq X$, where we agree that $\inf \emptyset = +\infty$*. Then, μ^* is an outer measure on X.*

It should be clear that, if there is at least one countable covering of E with elements of C, then the set $\{\sum_{j=1}^{+\infty}\tau(C_j)\,|\,C_1,C_2,\ldots\in\mathcal{C}$ so that $E\subseteq\cup_{j=1}^{+\infty}C_n\}$ is non-empty. If there is no countable covering of E with elements of C, then this set is empty and we take $\mu^*(E) = \inf\emptyset = +\infty$.

Proof: For $\emptyset$ the covering $\emptyset \subseteq \emptyset \cup \emptyset \cup \cdots$ implies $\mu^*(\emptyset) \leq \tau(\emptyset) + \tau(\emptyset) + \cdots = 0$ and, hence, $\mu^*(\emptyset) = 0$.

Now, let $A \subseteq B \subseteq X$. If there is no countable covering of B by elements of C, then $\mu^*(B) = +\infty$ and the inequality $\mu^*(A) \leq \mu^*(B)$ is obviously true. Otherwise, we take an arbitrary covering $B\subseteq\cup_{j=1}^{+\infty}C_n$ with $C_1,\ldots \in$ C. Then we also have $A\subseteq\cup_{j=1}^{+\infty}C_n$ and, by the definition of $\mu^*(A)$, we get $\mu^*(A) \leq$
$\sum_{j=1}^{+\infty}\tau(C_j)$). Taking the infimum of the right side, we find $\mu^*(A) \leq \mu^*(B)$.

Finally, let's prove $\mu^*(\cup_{n=1}^{+\infty}A_n) \leq \sum_{n=1}^{+\infty}\mu^*(A_n)$ for all $A_1,A_2,\ldots \subseteq X$. If the right side is $= +\infty$, the inequality is clear. Therefore we assume that the right side is $< +\infty$ and, hence, that $\mu^*(A_n) < +\infty$ for all n. By the definition of each $\mu^*(A_n)$, for every > 0 there exist $C_{n,1},C_{n,2},\ldots \in$ C so that $A_n\subseteq\cup_{j=1}^{+\infty}C_{n,j}$ and $\sum_{j=1}^{+\infty}\tau(C_{n,j})<\mu^*(A_n)+\frac{\epsilon}{2^n}$.

Then $\cup_{n=1}^{+\infty}A_n \subseteq \cup_{(n,j)\in\mathbf{N}\times\mathbf{N}}C_{n,j}$ and, using an arbitrary enumeration of $\mathbf{N}\times\mathbf{N}$ and Proposition 2.3, we get by the definition of $\mu^*(\cup_{n=1}^{+\infty}A_n)$ that $\mu^*(\cup_{n=1}^{+\infty}A_n) \leq \sum_{(n,j)\in\mathbf{N}\times\mathbf{N}}\tau(C_{n,j})$. Proposition 2.6 implies $\mu^*(\cup_{n=1}^{+\infty}A_n) \leq \sum_{n=1}^{+\infty}(\sum_{j=1}^{+\infty}\tau(C_{n,j})) < \sum_{n=1}^{+\infty}(\mu^*(A_n)+\frac{\epsilon}{2^n}) = \sum_{n=1}^{+\infty}\mu^*(A_n)+\epsilon$. Since is arbitrary, we conclude that $\mu^*(\cup_{n=1}^{+\infty}A_n)\leq\sum_{n=1}^{+\infty}\mu^*(A_n)$.

Check your progress

1. Let μ^* be an outer measure on X and $Y \subseteq X$. Define $\mu^*_Y(E) = \mu^*(E\cap Y)$ for all $E \subseteq X$ and prove that μ^*_Y is an outer measure on X and that Y is μ^*_Y-measurable.
2. Let μ^*,μ_1^*,μ_2^* be outer measures on X and $\kappa \in [0,+\infty)$. Prove that $\kappa\mu^*,\mu_1^*+\mu_2^*$ and $\max\{\mu_1^*,\mu_2^*\}$ are outer measures on X, where these are defined by the formulas

   $(\kappa\mu^*)(E) = \kappa\cdot\mu^*(E)$, $(\mu_1^*+\mu_2^*)(E)=\mu_1^*(E)+\mu_2^*(E)$ (consider $0 \cdot (+\infty) = 0$) and

   $\max\{\mu_1^*,\mu_2^*\}(E)=\max\{\mu_1^*(E),\mu_2^*(E)\}$

for all $E \subseteq X$.

Let X be a non-empty set and consider $\mu^*(\emptyset) = 0$ and $\mu^*(E) = 1$ if $\emptyset$ 6= $E \subseteq X$. Prove that μ^* is an outer measure on X and find all the μ^*-measurable subsets of X.

3.5 LET US SUM UP

In this unit we discussed the following ter measures

- Construction of outer measures

3.6 KEYWORDS

measures- measure on a **set** is a systematic way to assign a number to each suitable subset of that **set**, intuitively interpreted as its size. In this sense, a **measure** is a generalization of the concepts of length, area, and volume.

Set - **set** is a well-defined collection of distinct objects, considered as an object in its own right.

3.7 QUESTIONS FOR REVIEW

Let μ^* be an outer measure on X and $Y \subseteq X$. Define $\mu^*{}_Y(E) = \mu^*(E \cap Y)$ for all $E \subseteq X$ and prove that $\mu^*{}_Y$ is an outer measure on X and that Y is $\mu^*{}_Y$-measurable.

Let μ^*, μ_1^*, μ_2^* be outer measures on X and $\kappa \in [0,+\infty)$. Prove that $\kappa\mu^*, \mu_1^* + \mu_2^*$ and $\max\{\mu_1^*, \mu_2^*\}$ are outer measures on X, where these are defined by the formulas

$(\kappa\mu^*)(E) = \kappa \cdot \mu^*(E)$, $(\mu_1^* + \mu_2^*)(E) = \mu_1^*(E) + \mu_2^*(E)$ (consider $0 \cdot (+\infty) = 0$) and

$\max\{\mu_1^*, \mu_2^*\}(E) = \max\{\mu_1^*(E), \mu_2^*(E)\}$

for all $E \subseteq X$.

Let X be a non-empty set and consider $\mu^*(\emptyset) = 0$ and $\mu^*(E) = 1$ if $\emptyset$ 6= $E \subseteq X$. Prove that μ^* is an outer measure on X and find all the μ^*-measurable subsets of X.

For every $E \subseteq \mathbf{N}$ define $\kappa(E) = \limsup_{n\to+\infty} \frac{1}{n}$card($E\cap\{1,2,...,n\}$). Is κ an outer measure on **N**?

Let (μ_n^*) be a sequence of outer measures on X. Let $\mu^*(E) = \sup_n \mu_n^*(E)$ for all $E \subseteq X$ and prove that μ^* is an outer measure on X.

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3.9 ANSWERS TO CHECK YOUR PROGRESS

1.please check section 3.3-3.3.4

2.please check section 3.3-3.3.4

3.please check section 3.3-3.3.4

UNIT 4 LEBESGUE MEASURE ON $\mathbf{R}^N$

STRUCTURE

4.1 OBJECTIVES

In this chapter we are going to learn about volume of intervals , lebesgue measure and its transformations and also about cantor sets.

4.2 INTRODUCTION

L_n *is called the σ-algebra of Lebesgue sets in* $\mathbf{R}^n$, *(ii)* m_n^* *is called the (n-dimensional) Lebesgue outer measure on* $\mathbf{R}^n$ *and (iipai)* m_n *is called the (n-dimensional) Lebesgue measure on* $\mathbf{R}^n$.

Our aim now is to study properties of Lebesgue sets in $\mathbf{R}^n$ and especially their relation with the Borel sets or even more special sets in $\mathbf{R}^n$, like open sets or closed sets or unions of intervals.

4.3 VOLUME OF INTERVALS

We consider the function $\mathrm{vol}_n(S)$ defined for intervals S in $\mathbf{R}^n$, which is just the product of the lengths of the edges of S: the so-called **(*n*-**

dimensional) volume of S. In this section we shall investigate some properties of the volume of intervals.

Lemma 4.1 *Let* $P = (a_1,b_1] \times \cdots \times (a_n,b_n]$ *and, for each* $k = 1,\dots,n$, *let* $a_k = c_k^{(0)} < c_k^{(1)} < \cdots < c_k^{(m_k)} = b_k$. *If we set* $P_{i_1,\dots,i_n} = (c_1^{(i_1-1)}, c_1^{(i_1)}] \times \cdots \times (c_n^{(i_n-1)}, c_n^{(i_n)}]$ *for* $1 \le i_1 \le m_1,\dots,1 \le i_n \le m_n$, *then*

$$vol_n(P) = \sum_{1\le i_1\le m_1,\dots,1\le i_n\le m_n} vol_n(P_{i_1,\dots,i_n}).$$

Proof: For the second equality in the following calculation we use the distributive property of multiplication of sums:

$$\begin{aligned}
\sum_{1\le i_1\le m_1,\dots,1\le i_n\le m_n} \mathrm{vol}_n(P_{i_1,\dots,i_n})
&= \sum_{1\le i_1\le m_1,\dots,1\le i_n\le m_n} \left(c_1^{(i_1)} - c_1^{(i_1-1)}\right)\cdots\left(c_n^{(i_n)} - c_n^{(i_n-1)}\right)\\
&= \sum_{i_1=1}^{m_1}\left(c_1^{(i_1)} - c_1^{(i_1-1)}\right)\cdots \sum_{i_n=1}^{m_n}\left(c_n^{(i_n)} - c_n^{(i_n-1)}\right)\\
&= (b_1 - a_1)\cdots(b_n - a_n) = \mathrm{vol}_n(P).
\end{aligned}$$

Referring to the situation described by Lemma 4.1 we shall use the expression: *the intervals* $P_{i_1,\dots,i_n}$ *result from* P *by subdivision of its edges*.

Lemma 4.2 *Let* $P, P_1,\dots,P_l$ *be open-closed intervals and* $P_1,\dots,P_l$ *be pairwise disjoint. If* $P = P_1 \cup \cdots \cup P_l$, *then* $vol_n(P) = vol_n(P_1) + \cdots + vol_n(P_l)$.

Proof: Let $P = (a_1,b_1]\times\cdots\times(a_n,b_n]$ and $P_j = (a_1^{(j)}, b_1^{(j)}] \times \cdots \times (a_n^{(j)}, b_n^{(j)}]$ for every $j = 1,\dots,l$.

For every $k = 1,\dots,n$ we set

$$\{c_k^{(0)},\dots,c_k^{(m_k)}\} = \{a_k^{(1)},\dots,a_k^{(l)},b_k^{(1)},\dots,b_k^{(l)}\},$$

so that $a_k = c_k^{(0)} < c_k^{(1)} < \cdots < c_k^{(m_k)} = b_k$. This simply means that we *rename* the numbers $a_k^{(1)},\dots,a_k^{(l)},b_k^{(1)},\dots,b_k^{(l)}$ in increasing order and so that there are no repetitions. Of course, the smallest of these numbers is a_k and the largest is b_k, otherwise the $P_1,\dots,P_l$ would not cover P.

It is obvious that

every interval $(a_k^{(j)}, b_k^{(j)}]$ is the union of some successive among the intervals

$(c_k^{(0)}, c_k^{(1)}], \ldots, (c_k^{(m_k-1)}, c_k^{(m_k)}]$.

We now set

$P_{i_1,\ldots,i_n} = (c_1^{(i_1-1)}, c_1^{(i_1)}] \times \cdots \times (c_n^{(i_n-1)}, c_n^{(i_n)}]$

for $1 \le i_1 \le m_1, \ldots, 1 \le i_n \le m_n$.

It is clear that the $P_{i_1,\ldots,i_n}$'s result from P by subdivision of its edges. It is also almost clear that

the intervals among the $P_{i_1,\ldots,i_n}$ which belong to a P_j result from it by subdivision of its edges (this is due to i).

every $P_{i_1,\ldots,i_n}$ is included in exactly one from $P_1,\ldots,P_l$ (because the $P_1,\ldots,P_l$ are disjoint and cover P).

We now calculate, using Lemma 4.1 for the first and third equality and grouping together the intervals $P_{i_1,\ldots,i_n}$ which are included in the same P_j for the second equality:

$$\mathrm{vol}_n(P) = \sum_{1 \le i_1 \le m_1, \ldots, 1 \le i_n \le m_n} \mathrm{vol}_n(P_{i_1,\ldots,i_n})$$

$$= \sum_{j=1}^{l} \sum_{P_{i_1,\ldots,i_n} \subseteq P_j} \mathrm{vol}_n(P_{i_1,\ldots,i_n})$$

$$= \sum_{j=1}^{l} \mathrm{vol}_n(P_j).$$

Lemma 4.3 *Let $P, P_1, \ldots, P_l$ be open-closed intervals and $P_1, \ldots, P_l$ be pairwise disjoint. If $P_1 \cup \cdots \cup P_l \subseteq P$, then $vol_n(P_1) + \cdots + vol_n(P_l) \le vol_n(P)$.*

Proof: We know from Proposition 1.11 that $P \setminus (P_1 \cup \cdots \cup P_l) = P_1' \cup \cdots \cup P_k'$ for some pairwise disjoint open-closed intervals $P_1', \ldots, P_k'$. Then $P = P_1 \cup \cdots \cup P_l \cup P_1' \cup \cdots \cup P_k'$ and Lemma 4.2 now implies that $\mathrm{vol}_n(P) = \mathrm{vol}_n(P_1) + \cdots +$ $\mathrm{vol}_n(P_l) + \mathrm{vol}_n(P_1') + \cdots + \mathrm{vol}_n(P_k') \ge \mathrm{vol}_n(P_1) + \cdots + \mathrm{vol}_n(P_l)$.

Lemma 4.4 *Let $P, P_1, \ldots, P_l$ be open-closed intervals. If $P \subseteq P_1 \cup \cdots \cup P_l$, then $vol_n(P) \le vol_n(P_1) + \cdots + vol_n(P_l)$.*

Proof: We first write $P = P'_1 \cup \cdots \cup P'_l$ where $P'_j = P_j \cap P$ are open-closed intervals included in *P*. We then write $P = P'_1 \cup (P'_2 \setminus P'_1) \cup \cdots \cup \left(P'_l \setminus (P'_1 \cup \cdots \cup P'_{l-1})\right)$.

Each of these *l* pairwise disjoint sets can, by Proposition 1.11, be written as a finite union of pairwise disjoint open-closed intervals: $P'_1 = P'_1$ and

$P'_j \setminus (P'_1 \cup \cdots \cup P'_{j-1}) = P_1^{(j)} \cup \cdots \cup P_{m_j}^{(j)}$

for $2 \le j \le l$.

Lemma 4.2 for the equality and Lemma 4.3 for the two inequalities imply

$$\begin{aligned}\mathrm{vol}_n(P) &= \mathrm{vol}_n(P'_1) + \sum_{j=2}^{l}\Big(\sum_{m=1}^{m_j} \mathrm{vol}_n(P_m^{(j)})\Big)\\ &\le \mathrm{vol}_n(P'_1) + \sum_{j=2}^{l} \mathrm{vol}_n(P'_j) \le \sum_{j=1}^{l} \mathrm{vol}_n(P_j).\end{aligned}$$

Lemma 4.5 *Let Q be a closed interval and $R_1,\ldots,R_l$ be open intervals so that $Q \subseteq R_1 \cup \cdots \cup R_l$. Then $vol_n(Q) \le vol_n(R_1) + \cdots + vol_n(R_l)$.*

Proof: Let *P* and P_j be the open-closed intervals with the same edges as *Q* and, respectively, R_j. Then $P \subseteq Q \subseteq R_1 \cup \cdots \cup R_l \subseteq P_1 \cup \cdots \cup P_l$ and by Lemma 4.4, $\mathrm{vol}_n(Q) = \mathrm{vol}_n(P) \le \mathrm{vol}_n(P_1)+\cdots+\mathrm{vol}_n(P_l) = \mathrm{vol}_n(R_1)+\cdots+\mathrm{vol}_n(R_l)$.

4.4 LEBESGUE MEASURE IN R*N*.

Consider the collection C of all open intervals in $\mathbf{R}^n$ and the $\tau : C \to [0,+\infty]$ defined by $\tau(R) = \mathrm{vol}_n(R) = (b_1 - a_1)\cdots(b_n - a_n)$

for every $R = (a_1,b_1) \times \cdots \times (a_n,b_n) \in C$. If we define

$$m^*_n(E) = \inf\Big\{\sum_{j=1}^{+\infty} \mathrm{vol}_n(R_j) \,\Big|\, R_1,R_2,\ldots \in C \text{ so that } E \subseteq \cup_{j=1}^{+\infty} R_j\Big\}$$

for all $E \subseteq \mathbf{R}^n$, then Theorem 3.2 implies that m^*_n is an outer measure on $\mathbf{R}^n$.

We observe that, since $\mathbf{R}^n = \cup_{k=1}^{+\infty}(-k,k) \times \cdots \times (-k,k)$, for every $E \subseteq \mathbf{R}^n$ there is a countable covering of E by elements of C. Now Theorem 3.1 implies that the collection

$\mathcal{L}_n = \Sigma_{m_n^*}$

of m_n^*-measurable sets is a σ-algebra of subsets of $\mathbf{R}^n$ and, if m_n is defined as the restriction of m_n^* on $\mathcal{L}_n$, then m_n is a complete measure on (X, L_n).

Definition 4.1 *(i)* L_n *is called the σ-algebra of Lebesgue sets in* $\mathbf{R}^n$*, (ii)* m_n^* *is called the (n-dimensional) Lebesgue outer measure on* $\mathbf{R}^n$ *and (iipai)* m_n *is called the (n-dimensional) Lebesgue measure on* $\mathbf{R}^n$*.*

Our aim now is to study properties of Lebesgue sets in $\mathbf{R}^n$ and especially their relation with the Borel sets or even more special sets in $\mathbf{R}^n$, like open sets or closed sets or unions of intervals.

Theorem 4.1 *Every interval S in* $\mathbf{R}^n$ *is a Lebesgue set and*

$$m_n(S) = vol_n(S).$$

Proof: Let $Q = [a_1,b_1] \times \cdots \times [a_n,b_n]$.

Since $Q \subseteq (a_1 - \epsilon, b_1 + \epsilon) \times \cdots \times (a_n - \epsilon, b_n + \epsilon)$, we get by the definition of m_n^* that $m_n^*(Q) \leq \mathrm{vol}_n((a_1 - \epsilon, b_1 + \epsilon) \times \cdots \times (a_n - \epsilon, b_n + \epsilon)) = (b_1 - a_1 + 2\epsilon) \cdots (b_n - a_n + 2\epsilon)$. Since > 0 is arbitrary, we find $m_n^*(Q) \leq \mathrm{vol}_n(Q)$.

Now take any covering, $Q \subseteq R_1 \cup R_2 \cup \cdots$ of Q by open intervals. Since Q is compact, there is l so that $Q \subseteq R_1 \cup \cdots \cup R_l$ and Lemma 4.5 implies that $\mathrm{vol}_n(Q) \leq \mathrm{vol}_n(R_1) + \cdots + \mathrm{vol}_n(R_l) \leq \sum_{k=1}^{+\infty} \mathrm{vol}_n(R_k)$. Taking the infimum of the right side, we get $\mathrm{vol}_n(Q) \leq m_n^*(Q)$ and, hence,

$$m_n^*(Q) = \mathrm{vol}_n(Q).$$

Now take any general interval S and let $a_1,b_1,...,a_n,b_n$ be the end-points of its edges. Then $Q^0 \subseteq S \subseteq Q^{00}$, where $Q' = [a_1+\epsilon, b_1-\epsilon] \times \cdots \times [a_n+\epsilon, b_n-\epsilon]$ and $Q'' = [a_1 - \epsilon, b_1 + \epsilon] \times \cdots \times [a_n - \epsilon, b_n + \epsilon]$. Hence $m_n^*(Q') \leq m_n^*(S) \leq m_n^*(Q'')$, namely $(b_1 - a_1 - 2\epsilon) \cdots (b_n - a_n - 2\epsilon) \leq m_n^*(S) \leq (b_1 - a_1 + 2\epsilon) \cdots (b_n - a_n + 2\epsilon)$. Since > 0 is arbitrary, we find

$$m_n^*(S) = \mathrm{vol}_n(S).$$

Consider an open-closed interval P and an open interval R. Take the openclosed interval P_R with the same edges as R. Then $m_n^*(R \cap P) \leq m_n^*(P_R \cap P) =$

$\mathrm{vol}_n(P_R \cap P)$ and $m_n^*(R \cap P^c) \le m_n^*(P_R \cap P^c)$. Now Proposition 1.11 implies

$P_R \cap P^c = P_R \setminus P = P_1' \cup \cdots \cup P_k'$ for some pairwise disjoint open-closed intervals $P_1', \ldots, P_k'$. Hence $m_n^*(R \cap P^c) \le m_n^*(P_1') + \cdots + m_n^*(P_k') = \mathrm{vol}_n(P_1') + \cdots + \mathrm{vol}_n(P_k')$. Altogether, $m_n^*(R \cap P) + m_n^*(R \cap P^c) \le \mathrm{vol}_n(P_R \cap P) + \mathrm{vol}_n(P_1') + \cdots + \mathrm{vol}_n(P_k')$ and, by Lemma 4.2, this is = $\mathrm{vol}_n(P_R) = \mathrm{vol}_n(R)$. We have just proved that

$$m_n^*(R \cap P) + m_n^*(R \cap P^c) \le \mathrm{vol}_n(R).$$

Consider any open-closed interval P and any $E \subseteq \mathbf{R}^n$ with $m_n^*(E) < +\infty$. Take, for arbitrary $\epsilon > 0$, a covering $E \subseteq \cup_{j=1}^{+\infty} R_j$ of E by open intervals so that $\sum_{j=1}^{+\infty} \mathrm{vol}_n(R_j) < m_n^*(E) + \epsilon$. This implies $m_n^*(E \cap P) + m_n^*(E \cap P^c) \le \sum_{j=1}^{+\infty} m_n^*(R_j \cap P) + \sum_{j=1}^{+\infty} m_n^*(R_j \cap P^c) = \sum_{j=1}^{+\infty} [m_n^*(R_j \cap P) + m_n^*(R_j \cap P^c)]$ which, by the last result, is $\le \sum_{j=1}^{+\infty} \mathrm{vol}_n(R_j) < m_n^*(E) + \epsilon$. This implies that $m_n^*(E \cap P) + m_n^*(E \cap P^c) \le m_n^*(E)$ and P is a Lebesgue set.

If T is any interval at least one of whose edges is a single point, then $m^*_n(T) = \mathrm{vol}_n(T) = 0$ and, by Proposition 3.1, T is a Lebesgue set. Now, any interval S differs from the open-closed interval P, which has the same sides as S, by finitely many (at most $2n$) T's, and hence S is also a Lebesgue set.

Theorem 4.2 *Lebesgue measure is σ-finite but not finite.*

Proof: We write $\mathbf{R}^n = \cup_{k=1}^{+\infty} Q_k$ with $Q_k = [-k,k] \times \cdots \times [-k,k]$, where $m_n(Q_k) = \mathrm{vol}_n(Q_k) < +\infty$ for all k. On the other hand, for all k, $m_n(\mathbf{R}^n) \ge m_n(Q_k) = (2k)^n$ and, hence, $m_n(\mathbf{R}^n) = +\infty$.

Theorem 4.3 *All Borel sets in* $\mathbf{R}^n$ *are Lebesgue sets.*

Proof: Theorem 4.1 says that, if E is the collection of all intervals in $\mathbf{R}^n$, then E ⊆ L_n. But then $B_{\mathbf{R}^n} = \Sigma(E) \subseteq L_n$.

Therefore all open and all closed subsets of $\mathbf{R}^n$ are Lebesgue sets.

Theorem 4.4 *Let* $E \subseteq \mathbf{R}^n$. *Then*

$E \in \mathcal{L}_n$ *if and only if there is A, a countable intersection of open sets, so that* $E \subseteq A$ *and* $m_n^*(A \setminus E) = 0$.

$E \in \mathcal{L}_n$ *if and only if there is B, a countable union of compact sets, so that* $B \subseteq E$ *and* $m^*_n(E \setminus B) = 0$.

Proof: (i) One direction is easy. If there is A, a countable intersection of open sets, so that $E \subseteq A$ and $m_n^*(A \setminus E) = 0$, then, by Proposition 3.1, $A \setminus E \in \mathcal{L}_n$ and, thus, $E = A \setminus (A \setminus E) \in \mathcal{L}_n$.

To prove the other direction consider, after Theorem 4.2, $Y_1, Y_2, \ldots \in \mathcal{L}_n$ so that $\mathbf{R}^n = \cup_{k=1}^{+\infty} Y_k$ and $m_n(Y_k) < +\infty$ for all k. Define $E_k = E \cap Y_k$ so that $E = \cup_{k=1}^{+\infty} E_k$ and $m_n(E_k) < +\infty$ for all k.

For all k and arbitrary $l \in \mathbf{N}$ find a covering $E_k \subseteq \cup_{j=1}^{+\infty} R_j^{(k,l)}$ by open intervals so that $\sum_{j=1}^{+\infty} \mathrm{vol}_n(R_j^{(k,l)}) < m_n(E_k) + \frac{1}{l2^k}$ and set $U^{(k,l)} = \cup_{j=1}^{+\infty} R_j^{(k,l)}$. Then $E_k \subseteq U^{(k,l)}$ and $m_n(U^{(k,l)}) < m_n(E_k) + \frac{1}{l2^k}$, from which

$$m_n(U^{(k,l)} \setminus E_k) < \frac{1}{l2^k}.$$

Now set $U^{(l)} = \cup_{k=1}^{+\infty} U^{(k,l)}$. Then $U^{(l)}$ is open and $E \subseteq U^{(l)}$ and it is trivial to see that $U^{(l)} \setminus E \subseteq \cup_{k=1}^{+\infty}(U^{(k,l)} \setminus E_k)$, from which we get

$$m_n(U^{(l)} \setminus E) \le \sum_{k=1}^{+\infty} m_n(U^{(k,l)} \setminus E_k) < \sum_{k=1}^{+\infty} \frac{1}{l2^k} = \frac{1}{l}.$$

Finally, define $A = \cap_{l=1}^{+\infty} U^{(l)}$ to get $E \subseteq A$ and $m_n(A \setminus E) \le m_n(U^{(l)} \setminus E) < \frac{1}{l}$ for all l and, thus, $m_n(A \setminus E) = 0$.

(ii) If B is a countable union of compact sets so that $B \subseteq E$ and $m_n^*(E \setminus B) = 0$, then, by Proposition 3.1, $E \setminus B \in \mathcal{L}_n$ and thus $E = B \cup (E \setminus B) \in \mathcal{L}_n$.

Now take $E \in \mathcal{L}_n$. Then $E^c \in \mathcal{L}_n$ and by (i) there is an A, a countable intersection of open sets, so that $E^c \subseteq A$ and $m_n(A \setminus E^c) = 0$.

We set $B = A^c$, a countable union of closed sets, and we get $m_n(E \setminus B) = m_n(A \setminus E^c) = 0$. Now, let $B = \cup_{j=1}^{+\infty} F_j$, where each F_j is closed. We then write

$F_j = \cup_{k=1}^{+\infty} F_{j,k}$, where $F_{j,k} = F_j \cap ([-k,k] \times \cdots \times [-k,k])$ is a compact set. This proves that B is a countable union of compact sets: $B = \cup_{(j,k) \in \mathbf{N} \times \mathbf{N}} F_{j,k}$.

Theorem 4.4 says that *every Lebesgue set in* $\mathbf{R}^n$ *is, except from a null set, equal to a Borel set*.

Theorem 4.5 *(i) m_n is the only measure on* $(\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n})$ *with* $m_n(P) = vol_n(P)$ *for every open-closed interval P. (ii)* $(\mathbf{R}^n, \mathcal{L}_n, m_n)$ *is the completion of* $(\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, m_n)$.

Proof: (i) If μ is any measure on $(\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n})$ with $\mu(P) = \mathrm{vol}_n(P)$ for all open-closed intervals P, then it is trivial to see that $\mu(P) = +\infty$ for any unbounded generalised open-closed interval P: just take any increasing sequence of open-closed intervals having union P. Therefore $\mu(\cup_{j=1}^m P_j) = \sum_{j=1}^m \mu(P_j) = \sum_{j=1}^m m_n(P_j) = m_n(\cup_{j=1}^m P_j)$ for all pairwise disjoint open-closed generalised intervals $P_1,...,P_m$. Therefore the measures μ and m_n are equal on the algebra $\mathcal{A} = \{\cup_{j=1}^m P_j \,|\, m \in \mathbf{N}, P_1, \ldots, P_m$ pairwise disjoint open-closed generalised intervals$\}$. By Theorem 2.4, the two measures are equal also on $\Sigma(\mathcal{A}) = \mathcal{B}_{\mathbf{R}^n}$.

(ii) Let $(\mathbf{R}^n, \overline{\mathcal{B}_{\mathbf{R}^n}}, \overline{m_n})$ be the completion of $(\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, m_n)$.

By Theorem 4.3, $(\mathbf{R}^n, \mathcal{L}_n, m_n)$ is a complete extension of $(\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, m_n)$.

Hence, $\overline{\mathcal{B}_{\mathbf{R}^n}} \subseteq \mathcal{L}_n$ and $m_n(E) = \overline{m_n}(E)$ for every $E \in \overline{\mathcal{B}_{\mathbf{R}^n}}$.

Take any $E \in \mathcal{L}_n$ and, using Theorem 4.4, find a Borel set B so that $B \subseteq E$ and $m_n(E \setminus B) = 0$. Using Theorem 4.4 once more, find a Borel set A so that $(E \setminus B) \subseteq A$ and $m_n(A \setminus (E \setminus B)) = 0$. Therefore, $m_n(A) = m_n(A \setminus (E \setminus B)) + m_n(E \setminus B) = 0$.

Hence, we can write $E = B \cup L$, where $B \in \mathcal{B}_{\mathbf{R}^n}$ and $L = E \setminus B \subseteq A \in \mathcal{B}_{\mathbf{R}^n}$ with $m_n(A) = 0$. After Theorem 2.3, we see that E has the form of the typical element of $\overline{\mathcal{B}_{\mathbf{R}^n}}$ and, thus, $\mathcal{L}_n \subseteq \overline{\mathcal{B}_{\mathbf{R}^n}}$. This concludes the proof.

Theorem 4.6 *Suppose $E \in \mathcal{L}_n$ with $m_n(E) < +\infty$. For arbitrary $\epsilon > 0$, there are pairwise disjoint open intervals $R_1,...,R_l$ so that* $m_n(E \triangle (R_1 \cup \cdots \cup R_l)) < \epsilon$.

Proof: We consider a covering $E \subseteq \cup_{j=1}^{+\infty} R'_j$ by open intervals such that $\sum_{j=1}^{+\infty} \mathrm{vol}_n(R'_j) < m_n(E) + \frac{\epsilon}{2}$.

Now we consider the open-closed interval P'_j which has the same edges as

R_j^0, and then $E \subseteq \cup_{j=1}^{+\infty} P'_j$ and $\sum_{j=1}^{+\infty} \mathrm{vol}_n(P'_j) < m_n(E) + \frac{\epsilon}{2}$.

We take m so that $\sum_{j=m+1}^{+\infty} \mathrm{vol}_n(P'_j) < \frac{\epsilon}{2}$ and we observe the inclusions $E \setminus (P'_1 \cup \cdots \cup P'_m) \subseteq \cup_{j=m+1}^{+\infty} P'_j$ and $(P'_1 \cup \cdots \cup P'_m) \setminus E \subseteq \left(\cup_{j=1}^{+\infty} P'_j\right) \setminus E$. Thus,

$$m_n(E \setminus (P'_1 \cup \cdots \cup P'_m)) \leq \sum_{j=m+1}^{+\infty} \mathrm{vol}_n(P'_j) < \tfrac{\epsilon}{2} \text{ and}$$

$$m_n((P'_1 \cup \cdots \cup P'_m) \setminus E) \leq m_n(\cup_{j=1}^{+\infty} P'_j) - m_n(E) < \tfrac{\epsilon}{2}.$$ Adding, we find

$m_n(E \triangle (P'_1 \cup \cdots \cup P'_m)) < \epsilon.$

Proposition 1.11 implies that $P'_1 \cup \cdots \cup P'_m = P_1 \cup \cdots \cup P_l$ for some pairwise disjoint open-closed intervals $P_1, \cdots, P_l$ and, thus,

$m_n(E \triangle (P_1 \cup \cdots \cup P_l)) < \epsilon.$

We consider R_k to be the open interval with the same edges as P_k so that $\cup_{k=1}^{l} R_k \subseteq \cup_{k=1}^{l} P_k$ and

$m_n((\cup_{k=1}^{l} P_k) \setminus (\cup_{k=1}^{l} R_k)) \leq \sum_{k=1}^{l} m_n(P_k \setminus R_k) = 0.$

This, easily, implies that

$m_n(E \triangle (R_1 \cup \cdots \cup R_l)) = m_n(E \triangle (P_1 \cup \cdots \cup P_l)) < \epsilon.$

4.5 LEBESGUE MEASURE AND SIMPLE TRANSFORMATIONS.

Some of the simplest and most important transformations of $\mathbf{R}^n$ are the translations and the linear transformations.

Every $y \in \mathbf{R}^n$ defines the **translation** $\tau_y : \mathbf{R}^n \to \mathbf{R}^n$ by the formula

$$\tau_y(x) = x + y, \; x \in \mathbf{R}^n.$$

Then τ_y is an one-to-one transformation of $\mathbf{R}^n$ onto $\mathbf{R}^n$ and its inverse transformation is τ_{-y}. For every $E \subseteq \mathbf{R}^n$ we define

$$y + E = \{y + x | x \in E\} (= \tau_y(E)).$$

Every $\lambda > 0$ defines the **dilation** $l_\lambda : \mathbf{R}^n \to \mathbf{R}^n$ by the formula

$$l_\lambda(x) = \lambda x, \; x \in \mathbf{R}^n.$$

Then l_λ is an one-to-one transformation of $\mathbf{R}^n$ onto $\mathbf{R}^n$ and its inverse transformation is $l_{\frac{1}{\lambda}}$. For every $E \subseteq \mathbf{R}^n$ we define

$\lambda E = \{\lambda x | x \in E\} (= l_\lambda(E)).$

If S is any interval in $\mathbf{R}$, then any translation transforms it onto another interval (of the same

$(b_1, \ldots, y_n + a_n, y_n + b_n)$ $($ $)$

type) with the same volume. In fact, if $a_1,b_1,...,a_n,b_n$ are the end-points of the edges of S, then the translated $y+S$ has y_1+a_1,y_1+ as end-points of its edges. Therefore $\mathrm{vol}_n(y+S)=$

$(y_1+b_1)-(y_1+a_1)\cdots(y_n+b_n)-(y_n+a_n)=(b_1-a_1)\cdots(b_n-a_n)=\mathrm{vol}_n(S)$.

If we dilate the interval S with $a_1,b_1,...,a_n,b_n$ as end-points of its edges by the number $\lambda>0$, then we get the interval λS with $\lambda a_1,\lambda b_1,...,\lambda a_n,\lambda b_n$ as end-points of its edges. Therefore, $\mathrm{vol}_n(\lambda S)=(\lambda b_1-\lambda a_1)\cdots(\lambda b_n-\lambda a_n)=\lambda^n(b_1-a_1)\cdots(b_n-a_n)=\lambda^n\mathrm{vol}_n(S)$.

Another transformation is r, **reflection through** 0, with the formula

$$r(x)=-x,\ x\in\mathbf{R}^n.$$

This is one-to-one onto $\mathbf{R}^n$ and it is the inverse of itself. We define

$$-E=\{-x|x\in E\}(=r(E))$$

for all $E\subseteq\mathbf{R}^n$. If S is any interval with $a_1,b_1,...,a_n,b_n$ as end-points of its edges, then $-S$ is an interval with $-b_1,-a_1,...,-b_n,-a_n$ as end-points of its edges and $\mathrm{vol}_n(-S)=(-a_1+b_1)\cdots(-a_n+b_n)=\mathrm{vol}_n(S)$.

After all these, we may say that *n-dimensional volume of intervals is invariant under translations and reflection* and *it is positive-homogeneous of degree n under dilations*.

We shall see that the same are true for n-dimensional Lebesgue measure of Lebesgue sets in $\mathbf{R}^n$.

Theorem 4.7 *(i)* L_n *is invariant under translations, reflection and dilations. That is, for all* $A\in\mathrm{L}_n$ *we have that* $y+A,-A,\lambda A\in\mathrm{L}_n$ *for every* $y\in\mathbf{R}^n,\lambda>0$.

(ii) m_n *is invariant under translations and reflection and positive-homogeneous of degree n under dilations. That is, for all* $A\in\mathrm{L}_n$ *we have that* $m_n(y+A)=m_n(A)$, $m_n(-A)=m_n(A)$, $m_n(\lambda A)=\lambda^n m_n(A)$ *for every* $y\in\mathbf{R}^n,\lambda>0$.

Proof: Let $E\subseteq\mathbf{R}^n$ and $y\in\mathbf{R}^n$. Then for all coverings $E\subseteq\cup_{j=1}^{+\infty}R_j$ by open intervals we get $y+E\subseteq\cup_{j=1}^{+\infty}(y+R_j)$. Therefore, $m_n^*(y+E)\leq\sum_{j=1}^{+\infty}\mathrm{vol}_n(y+R_j)=\sum_{j=1}^{+\infty}\mathrm{vol}_n(R_j)$. Taking the infimum of the right side, we find that $m_n^*(y+E)\leq m_n^*(E)$. Now, applying this to $y+E$ translated by $-y$, we get $m_n^*(E)=m_n^*(-y+(y+E))\leq m_n^*(y+E)$. Hence

$m_n^*(y+E)=m_n^*(E)$

for all $E \subseteq \mathbf{R}^n$ and $y \in \mathbf{R}^n$.

Similarly, $-E \subseteq \cup_{j=1}^{+\infty}(-R_j)$, which implies $m_n^*(-E) \leq \sum_{j=1}^{+\infty} \mathrm{vol}_n(-R_j) = \sum_{j=1}^{+\infty} \mathrm{vol}_n(R_j)$. Hence $m_n^*(-E) \leq m_n^*(E)$. Applying this to $-E$, we also get $m_n^*(E) = m_n^*(-(-E)) \leq m_n^*(-E)$ and, thus,

$m_n^*(-E) = m_n^*(E)$

for all $E \subseteq \mathbf{R}^n$.

Also, $\lambda E \subseteq \cup_{j=1}^{+\infty}(\lambda R_j)$, from which we get $m_n^*(\lambda E) \leq \sum_{j=1}^{+\infty} \mathrm{vol}_n(\lambda R_j) = \lambda^n \sum_{j=1}^{+\infty} \mathrm{vol}_n(R_j)$ and hence $m_n^*(\lambda E) \leq \lambda^n m_n^*(E)$. Applying to $\frac{1}{\lambda}$ and to λE, we find $m_n^*(E) = m_n^*(\frac{1}{\lambda}(\lambda E)) \leq (\frac{1}{\lambda})^n m_n^*(\lambda E)$, which gives

$$m_n^*(\lambda E) = \lambda^n m_n^*(E).$$

Suppose now that $A \in \mathcal{L}_n$ and $E \subseteq \mathbf{R}^n$.

We have $m_n^*(E \cap (y + A)) + m_n^*(E \cap (y + A)^c) = m_n^*\big(y + [(-y + E) \cap A]\big) + m_n^*\big(y + [(-y + E) \cap A^c]\big) = m_n^*\big((-y + E) \cap A\big) + m_n^*\big((-y + E) \cap A^c\big) = m_n^*(-y + E) = m_n^*(E)$. Therefore, $y + A \in \mathcal{L}_n$.

In the same way, $m_n^*(E \cap (-A)) + m_n^*(E \cap (-A)^c) = m_n^*\big(-[(-E) \cap A]\big) + m_n^*\big(-[(-E) \cap A^c]\big) = m_n^*\big((-E) \cap A\big) + m_n^*\big((-E) \cap A^c\big) = m_n^*(-E) = m_n^*(E)$. Therefore, $-A \in \mathcal{L}_n$. We, finally, have $m_n^*(E \cap (\lambda A)) + m_n^*(E \cap (\lambda A)^c) = m_n^*\big(\lambda[(\frac{1}{\lambda}E) \cap A]\big) + m_n^*\big(\lambda[(\frac{1}{\lambda}E) \cap A^c]\big) = \lambda^n m_n^*\big((\frac{1}{\lambda}E) \cap A\big) + \lambda^n m_n^*\big((\frac{1}{\lambda}E) \cap A^c\big) = \lambda^n m_n^*(\frac{1}{\lambda}E) = m_n^*(E)$. Therefore, $\lambda A \in \mathcal{L}_n$.

If $A \in \mathcal{L}_n$, then $m_n(y + A) = m_n^*(y + A) = m_n^*(A) = m_n(A)$, $m_n(-A) = m_n^*(-A) = m_n^*(A) = m_n(A)$ and $m_n(\lambda A) = m_n^*(\lambda A) = \lambda^n m_n^*(A) = \lambda^n m_n(A)$.

Reflection and dilations are special cases of linear transformations of $\mathbf{R}^n$. As is well known, a *linear transformation of* $\mathbf{R}^n$ is a function $T : \mathbf{R}^n \to \mathbf{R}^n$ such that

$T(x + y) = T(x) + T(y), T(\kappa x) = \kappa T(x), \quad x, y \in \mathbf{R}^n, \kappa \in \mathbf{R},$

and every such T has a *determinant*, $\det(T) \in \mathbf{R}$. In particular, $\det(r) = (-1)^n$ and $\det(l_\lambda) = \lambda^n$.

We recall that a linear transfomation T of $\mathbf{R}^n$ is one-to-one and onto $\mathbf{R}^n$ if and only if $\det(T)$ 6= 0. Moreover, if $\det(T)$ 6= 0, then T^{-1} is also a linear transformation of $\mathbf{R}^n$ and $\det(T^{-1}) = (\det(T))^{-1}$. Finally, if T,T_1,T_2 are linear transformations of $\mathbf{R}^n$ and $T = T_1 \circ T_2$, then $\det(T) = \det(T_1)\det(T_2)$.

Theorem 4.8 *Let* $T : \mathbf{R}^n \to \mathbf{R}^n$ *be a linear transformation. If* $A \in \mathcal{L}_n$, *then* $T(A) \in \mathcal{L}_n$ *and* $m_n(T(A)) = |\det(T)|m_n(A)$.

If $|\det(T)| = 0$ *and* $m_n(A) = +\infty$, *we interpret the right side as* $0 \cdot (+\infty) = 0$.

Proof: At first we assume that $\det(T)$ 6= 0.

If T has the form $T(x_1,x_2,...,x_n) = (\lambda x_1,x_2,...,x_n)$ for a certain $\lambda \in \mathbf{R} \setminus \{0\}$, then $\det(T) = \lambda$ and, if $P = (a_1,b_1] \times (a_2,b_2] \times \cdots \times (a_n,b_n]$, then $T(P) = (\lambda a_1,\lambda b_1]\times(a_2,b_2]\times\cdots\times(a_n,b_n]$ or $T(P) = [\lambda b_1,\lambda a_1)\times(a_2,b_2]\times\cdots\times (a_n,b_n]$, depending on whether $\lambda > 0$ or $\lambda < 0$. Thus $T(P)$ is an interval and $m_n(T(P)) = |\lambda|m_n(P) = |\det(T)|m_n(P)$.

If $T(x_1,x_2,...,x_{i-1},x_i,x_{i+1},...,x_n) = (x_i,x_2,...,x_{i-1},x_1,x_{i+1},...,x_n)$ for a certain i 6= 1, then $\det(T) = -1$ and, if $P = (a_1,b_1] \times (a_2,b_2] \times \cdots \times (a_{i-1},b_{i-1}]\times(a_i,b_i]\times(a_{i+1},b_{i+1}]\times\cdots\times(a_n,b_n]$, then $T(P) = (a_i,b_i]\times(a_2,b_2]\times \cdots\times(a_{i-1},b_{i-1}]\times(a_1,b_1]\times(a_{i+1},b_{i+1}]\times\cdots\times(a_n,b_n]$. Thus $T(P)$ is an interval and $m_n(T(P)) = m_n(P) = |\det(T)|m_n(P)$.

If $T(x_1,...,x_{i-1},x_i,x_{i+1},...,x_n) = (x_1,...,x_{i-1},x_i + x_1,x_{i+1},...,x_n)$ for a certain i 6= 1, then $\det(T) = 1$ and, if $P = (a_1,b_1] \times \cdots \times (a_{i-1},b_{i-1}] \times (a_i,b_i] \times (a_{i+1},b_{i+1}] \times \cdots \times (a_n,b_n]$, then $T(P)$ is not an interval any more but $T(P) = \{(y_1,...,y_n)|y_j \in (a_j,b_j]$ for j 6= $i, y_i - y_1 \in (a_i,b_i]\}$ is a Borel set and hence it is in $\mathcal{L}_n$. We define the following three auxilliary sets: $L = (a_1,b_1] \times \cdots \times (a_{i-1},b_{i-1}] \times (a_i + a_1,b_i + b_1] \times (a_{i+1},b_{i+1}] \times \cdots \times (a_n,b_n]$, $M = \{(y_1,...,y_n)|y_j \in (a_j,b_j]$ for j 6= $i, a_i + a_1 < y_i \le a_i + y_1\}$ and $N = \{(y_1,...,y_n)|y_j \in (a_j,b_j]$ for j 6= $i, b_i + a_1 < y_i \le b_i + y_1\}$. It is easy to see that all four sets, $T(P),L,M,N$, are Borel sets and $T(P) \cap M = \emptyset$, $L \cap N = \emptyset$, $T(P)\cup M = L\cup N$ and that $N = x_0 + M$, where $x_0 = (0,...,0,b_i - a_i,0,...,0)$. Then $m_n(T(P)) + m_n(M) = m_n(L) + m_n(N)$ and $m_n(M) = m_n(N)$, implying that $m_n(T(P)) = m_n(L) = m_n(P) = |\det(T)|m_n(P)$, because L is an interval.

Now, let T be any linear transformation of the above three types. We have shown that $m_n(T(P)) = |\det(T)|m_n(P)$

for every open-closed interval P. If $R = (a_1,b_1) \times \cdots \times (a_n,b_n)$ it is easy to see, just as in the case of open-closed intervals, that $T(R)$ is a Borel set. We consider $P_1 = (a_1,b_1] \times \cdots \times (a_n,b_n]$ and $P_2 = (a_1,b_1-\epsilon] \times \cdots \times (a_n,b_n-\epsilon]$ and, from $P_2 \subseteq R \subseteq P_1$ we get $T(P_2) \subseteq T(R) \subseteq T(P_1)$. Hence $|\det(T)|m_n(P_2) \leq m_n(T(R)) \leq |\det(T)|m_n(P_1) = |\det(T)|m_n(R)$ and, taking the limit as $\epsilon \to 0+$, we find $m_n(T(R)) = |\det(T)|m_n(R)$

for every open interval R.

Let, again, T be any linear transformation of one of the above three types. Take any $E \subseteq \mathbf{R}^n$ and consider an arbitrary covering $E \subset \cup_{j=1}^{+\infty} R_j$ by open intervals. Then $T(E) \subseteq \cup_{j=1}^{+\infty} T(R_j)$ and hence $m_n^*(T(E)) \leq \sum_{j=1}^{+\infty} m_n(T(R_j)) = |\det(T)| \sum_{j=1}^{+\infty} m_n(R_j)$. Taking the infimum over all coverings, we conclude

$$m_n^*(T(E)) \leq |\det(T)| m_n^*(E).$$

If T is any linear transformation with $\det(T) \neq 0$, by a well-known result of Linear Algebra, there are linear transformations $T_1,\ldots,T_N$, where each is of one of the above three types so that $T = T_1 \circ \cdots \circ T_N$. Applying the last result repeatedly, we find $m_n^*(T(E)) \leq |\det(T_1)| \cdots |\det(T_N)| m_n^*(E)| = |\det(T)| m_n^*(E)$ for every $E \subseteq \mathbf{R}^n$. In this inequality, use now the set $T(E)$ in the place of E and T^{-1} in the place of T, and get $m_n^*(E) \leq |\det(T^{-1})| m_n^*(T(E)) = |\det(T)|^{-1} m_n^*(T(E))$. Combining the two inequalities, we conclude that

$$m_n^*(T(E)) = |\det(T)| m_n^*(E)$$

for every linear transformation T with $\det(T) \neq 0$ and every $E \subseteq \mathbf{R}^n$.

Let $A \in \mathcal{L}_n$. For all $E \subseteq \mathbf{R}^n$ we get

$$m_n^*(E \cap T(A)) + m_n^*(E \cap (T(A))^c) = m_n^*\big(T(T^{-1}(E) \cap A)\big) + m_n^*\big(T(T^{-1}(E) \cap A^c)\big) = |\det(T)|[m_n^*(T^{-1}(E) \cap A) + m_n^*(T^{-1}(E) \cap A^c)] = |\det(T)| m_n^*(T^{-1}(E)) = m_n^*(E).$$

This says that $T(A) \in \mathcal{L}_n$. Moreover,

$$m_n(T(A)) = m_n^*(T(A)) = |\det(T)| m_n^*(A) = |\det(T)| m_n(A).$$

If $\det(T) = 0$, then $V = T(\mathbf{R}^n)$ is a linear subspace of $\mathbf{R}^n$ with $\dim(V) \leq n-1$. We shall prove that $m_n(V) = 0$ and, from the completeness of m_n, we shall conclude that $T(A) \subseteq V$ is in $\mathcal{L}_n$ with $m_n(T(A)) = 0 = |\det(T)| m_n(A)$ for every $A \in \mathcal{L}_n$.

Let $\{f_1,...,f_m\}$ be a base of V (with $m \le n-1$) and complete it to a base $\{f_1,...,f_m,f_{m+1},...,f_n\}$ of $\mathbf{R}^n$. Take the linear transformation $S : \mathbf{R}^n \to \mathbf{R}^n$ given by

$$S(x_1 f_1 + \cdots + x_n f_n) = (x_1,...,x_n).$$

Then S is one-to-one and, hence, det(S) 6= 0. Moreover

$$S(V) = \{(x_1,...,x_m,0,...,0) | x_1,...,x_m \in \mathbf{R}\}.$$

We have $S(V) = \cup_{k=1}^{+\infty} Q_k$, where $Q_k = [-k,k]\times\cdots\times[-k,k]\times\{0\}\times\cdots\times\{0\}$. Each Q_k is a closed interval in $\mathbf{R}^n$ with $m_n(Q_k) = 0$. Hence, $m_n(S(V)) = 0$ and, then, $m_n(V) = |\det(S)|^{-1} m_n(S(V)) = 0$.

If $b,b_1,...,b_n \in \mathbf{R}^n$, then the set

$$M = \{b + \kappa_1 b_1 + \cdots + \kappa_n b_n \,|\, 0 \le \kappa_1,...,\kappa_n \le 1\}$$

is the typical *closed parallelepiped* in $\mathbf{R}^n$. One of the vertices of M is b and $b_1,...,b_n$ (interpreted as vectors) are the edges of M which start from b. For such an M we define the linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ by $T(x) = T(x_1,...,x_n) = x_1 b_1 + \cdots + x_n b_n$ for every $x = (x_1,...,x_n) \in \mathbf{R}^n$. We also consider the translation τ_b and observe that

$$M = \tau_b\big(T(Q_0)\big),$$

where $Q_0 = [0,1]^n$ is the unit qube in $\mathbf{R}^n$. Theorems 4.7 and 4.8 imply that M is a Lebesgue set and

$$m_n(M) = m_n\big(T(Q_0)\big) = |\det(T)| m_n(Q_0) = |\det(T)|.$$

The matrix of T with respect to the standard basis $\{e_1,...,e_n\}$ of $\mathbf{R}^n$ has as columns the vectors $T(e_1) = b_1,...,T(e_n) = b_n$. We conclude with the rule that *the Lebesgue measure of a closed parallelepiped is given by the absolute value of the determinant of the matrix having as columns the sides of the parallelepiped starting from one of its vertices.* Of course, it is easy to see that the same is true for any parallelepiped.

4.6 CANTOR SET.

Since $\{x\}$ is a degenerate interval, we see that $m_n(\{x\}) = \mathrm{vol}_n(\{x\}) = 0$. In fact, every countable subset of $\mathbf{R}^n$ has Lebesgue measure zero: if $A = \{x_1,x_2,...\}$, then $m_n(A) = \sum_{k=1}^{+\infty} m_n(\{x_k\}) = 0.$

The aim of this section is to provide an uncountable set in **R** whose Lebesgue measure is zero.

We start with the interval

$I_0 = [0,1]$,

then take $$I_1 = \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right],$$

next $$I_2 = \left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{1}{3}\right] \cup \left[\frac{2}{3},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right]$$,

and so on, each time dividing each of the intervals we get at the previous stage into three subintervals of equal length and keeping only the two closed subintervals on the sides.

Therefore, we construct a decreasing sequence (I_n) of closed sets so that every I_n consists of 2^n closed intervals all of which have the same length $\frac{1}{3^n}$. We define

$C = \cap_{n=1}^{+\infty} I_n$

and call it **the Cantor set**.

C is a compact subset of $[0,1]$ with $m_1(C) = 0$. To see this observe that for every n, $m_1(C) \leq m_1(I_n) = 2^n \cdot \frac{1}{3^n}$ which tends to 0 as $n \to +\infty$.

We shall prove by contradiction that C is uncountable. Namely, assume that $C = \{x_1,x_2,...\}$. We shall describe an inductive process of picking one from the subintervals constituting each I_n.

It is obvious that every x_n belongs to I_n, since it belongs to C. At the first step choose the interval $I^{(1)}$ to be the subinterval of I_1 which does *not* contain x_1. Now, $I^{(1)}$ includes two subintervals of I_2 and at the second step choose the interval $I^{(2)}$ to be whichever of these two subintervals of $I^{(1)}$ does *not* contain x_2. (If both do not contain x_2, just take the left one.) And continue inductively: if you have already chosen $I^{(n-1)}$ from the subintervals of I_{n-1}, then this includes two subintervals of I_n. Choose as $I^{(n)}$ whichever of these two subintervals of

$I^{(n-1)}$ does *not* contain x_n. (If both do not contain x_n, just take the left one.) This produces a sequence $(I^{(n)})$ of intervals with the following properties:

$I^{(n)} \subseteq I_n$ for all n,

$I(n) \subseteq I(n-1)$ for all n,

vol $_1(I^{(n)}) = \frac{1}{3^n} \to 0$ and

$x_n \notin I^{(n)}$ for all n.

From (ii) and (iii) we conclude that the intersection of all $I^{(n)}$'s contains a single point:

$\cap_{n=1}^{+\infty} I^{(n)} = \{x_0\}$

for some x_0. From (i) we see that $x_0 \in I_n$ for all n and thus $x_0 \in C$. Therefore, $x_0 = x_n$ for some $n \in \mathbf{N}$. But then $x_0 \in I^{(n)}$ and, by (iv), *the same* point x_n does not belong to $I^{(n)}$.

We get a contradiction and, hence, C is uncountable.

4.7 A NON-LEBESGUE SET IN R.

We consider the following equivalence relation in the set $[0,1)$. For any $x,y \in [0,1)$ we write $x \sim y$ if and only if $x-y \in \mathbf{Q}$. That $\sim$ is an equivalence relation is easy to see:

$x \sim x$, because $x - x = 0 \in \mathbf{Q}$.

If $x \sim y$, then $x - y \in \mathbf{Q}$, then $y - x = -(x - y) \in \mathbf{Q}$, then $y \sim x$. (c) If $x \sim y$ and $y \sim z$, then $x - y \in \mathbf{Q}$ and $y - z \in \mathbf{Q}$, then $x - z = (x - y) + (y - z) \in \mathbf{Q}$, then $x \sim z$.

Using the Axiom of Choice, we form a set N containing exactly one element from each equivalence class of $\sim$. This means that:

for every $x \in [0,1)$ there is exactly one $\bar{x} \in N$ so that $x - \bar{x} \in \mathbf{Q}$.

Indeed, if we consider the equivalence class of x and the element $\bar{x}$ of N from this equivalence class, then $\bar{x} \sim x$ and hence $\bar{x} - x \in \mathbf{Q}$. Moreover, if there are two $\bar{x},\bar{\bar{x}} \in N$ so that $x-\bar{x} \in \mathbf{Q}$ and $x-\bar{\bar{x}} \in \mathbf{Q}$, then $x \sim \bar{x}$ and $x \sim \bar{\bar{x}}$, implying that N contains two different elements from the equivalence class of x.

Our aim is to prove that N is not a Lebesgue set. We form the set

$$A = \cup_{r \in \mathbf{Q} \cap [0,1)} (N + r).$$

Diferent $(N + r)'$s are disjoint:

if $r_1, r_2 \in \mathbf{Q} \cap [0,1)$ and $r_1 \neq r_2$, then $(N + r_1) \cap (N + r_2) = \emptyset$.

Indeed, if $x \in (N + r_1) \cap (N + r_2)$, then $x - r_1, x - r_2 \in N$. But $x \sim x - r_1$ and $x \sim x - r_2$, implying that N contains two different (since $r_1 \neq r_2$) elements from the equivalence class of x.

$A \subseteq [0,2)$.

This is clear, since $N \subseteq [0,1)$ implies $N + r \subseteq [0,2)$ for all $r \in \mathbf{Q} \cap [0,1)$.

Take an arbitrary $x \in [0,1)$ and, by (i), the unique $\bar{x} \in N$ with $x - \bar{x} \in \mathbf{Q}$. Since $-1 < x - \bar{x} < 1$ we consider cases: if $r = x - \bar{x} \in [0,1)$, then $x = \bar{x} + r \in N+r \subseteq A$, while if $r = x-\bar{x} \in (-1,0)$, then $x+1 = \bar{x}+(r+1) \in N+(r+1) \subseteq A$. Therefore, for every $x \in [0,1)$ either $x \in A$ or $x + 1 \in A$. It is easy to see that exactly one of these two cases is true. Because if $x \in A$ and $x + 1 \in A$, then $x \in N + r_1$ and $x + 1 \in N + r_2$ for some $r_1,r_2 \in \mathbf{Q} \cap [0,1)$. Hence, $x-r_1, x+1-r_2 \in N$ and N contains two different (since $r_2 - r_1 \neq 1$) elements of the equivalence class of x. Thus, if we define the sets

$$E_1 = \{x \in [0,1) | x \in A\},\ E_2 = \{x \in [0,1) | x + 1 \in A\}$$

then we have proved that

$E_1 \cup E_2 = [0,1)$, $E_1 \cap E_2 = \emptyset$.

From (iv) we shall need only that $[0,1) \subseteq E_1 \cup E_2$.

We can also prove that

$E_1 \cup (E_2 + 1) = A$, $E_1 \cap (E_2 + 1) = \emptyset$.

In fact, the second is easy because $E_1,E_2 \subseteq [0,1)$ and hence $E_2 + 1 \subseteq [1,2)$. The first is also easy. If $x \in E_1$ then $x \in A$. If $x \in E_2 + 1$ then $x - 1 \in E_2$ and then $x = (x - 1) + 1 \in A$. Thus $E_1 \cup (E_2 + 1) \subseteq A$. On the other hand, if $x \in A \subseteq [0,2)$, then, either $x \in A \cap [0,1)$ implying $x \in E_1$, or $x \in A \cap [1,2)$ implying $x - 1 \in E_2$ i.e. $x \in E_2 + 1$. Thus $A \subseteq E_1 \cup (E_2 + 1)$.

From (v) we shall need only that $E_1, E_2 + 1 \subseteq A$.

Suppose N is a Lebesgue set. By (ii) and by the invariance of m_1 under translations, we get that $m_1(A) = \sum_{r\in\mathbf{Q}\cap[0,1)} m_1(N + r) = \sum_{r\in\mathbf{Q}\cap[0,1)} m_1(N)$. If $m_1(N) > 0$, then $m_1(A) = +\infty$, contradicting (iii). If $m_1(N) = 0$, then $m_1(A) = 0$, implying by (v) that $m_1(E_1) = m_1(E_2 + 1) = 0$, hence $m_1(E_1) = m_1(E_2) = 0$, and finally from (iv), $1 = m_1([0,1)) \leq m_1(E_1) + m_1(E_2) = 0$.

We arrive at a contradiction and N is not a Lebesgue set.

Check your progress

1.If $A \in \mathcal{L}_n$ and A is bounded, prove that $m_n(A) < +\infty$. Give an example of an $A \in \mathcal{L}_n$ which is not bounded but has $m_n(A) < +\infty$.

The invariance of Lebesgue measure under isometries.

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be an isometric linear transformation. This means that T is a linear transformation satisfying $|T(x) - T(y)| = |x - y|$ for every $x,y \in$

$\mathbf{R}^n$ or, equivalently, $TT^* = T^* T = I$, where T^* is the adjoint of T and I is the identity transformation.

2.Prove that, for every $E \in \mathrm{L}_n$, we have $m_n(T(E)) = m_n(E)$.

3.A parallelepiped in $\mathbf{R}^n$ is called **degenerate** if it is included in a hyperplane of $\mathbf{R}^n$, i.e. in a set of the form $b + V$, where $b \in \mathbf{R}^n$ and V is a linear subspace of $\mathbf{R}^n$ with $\dim(V) = n - 1$.

Prove that a parallelepiped M is degenerate if and only if $m_n(M) = 0$.

State in a formal way and prove the rule

volume = base area × height

for parallelepipeds in $\mathbf{R}^n$.

Regularity of Lebesgue measure.

Suppose that $A \in \mathrm{L}_n$.

4.Prove that there is a decreasing sequence (U_j) of open sets in $\mathbf{R}^n$ so that $A \subseteq U_j$ for all j and $m_n(U_j) \to m_n(A)$. Conclude that

$m_n(A) = \inf\{m_n(U) | U \text{ open} \supseteq A\}$.

4.8 LET US SUM UP

In this unit we discussed the following

Volume of intervals.

Lebesgue measure in $\mathbf{R}^n$.

Lebesgue measure and simple transformations

Cantor set

A non-Lebesgue set in $\mathbf{R}$

4.9 KEYWORDS

Cantor set- **Cantor set** is a **set** of points lying on a single line segment that has a number of remarkable and deep properties. It was discovered in 1874 by Henry John Stephen Smith and introduced by German mathematician Georg **Cantor** in 1883.

Lebesgue measure-In **measure** theory, a branch of mathematics, the **Lebesgue measure**, named after French mathematician Henri **Lebesgue**, is the standard way of assigning a **measure** to subsets of n-dimensional Euclidean space. For n = 1, 2, or 3, it coincides with the standard **measure** of length, area, or volume.

4.10 QUESTIONS FOR REVIEW

Prove that there is an increasing sequence (K_j) of compact sets in $\mathbf{R}^n$ so that $K_j \subseteq A$ for all j and $m_n(K_j) \to m_n(A)$. Conclude that $m_n(A) = \sup\{m_n(K) | K \text{ compact} \subseteq A\}$.

The validity of (i) and (ii) for $(\mathbf{R}^n, \mathcal{L}_n, m_n)$ is called **regularity**. We shall study this notion in chapter 5.

An example of an m_1-null uncountable set, dense in an interval. Let $\mathbf{Q} \cap [0,1] = \{x_1, x_2, ...\}$. For every > 0 we define

$$U(\epsilon) = \cup_{j=1}^{+\infty}\Big(x_j - \frac{\epsilon}{2^j}, x_j + \frac{\epsilon}{2^j}\Big), \qquad A = \cap_{n=1}^{+\infty} U\Big(\frac{1}{n}\Big).$$

Prove that $m_1(U(\epsilon)) \leq 2\epsilon$.

If $\epsilon < \frac{1}{2}$, prove that [0,1] is not a subset of $U(\epsilon)$.(iii) Prove that $A \subseteq [0,1]$ and $m_1(A) = 0$.

(iv) Prove that $\mathbf{Q} \cap [0,1] \subseteq A$ and that A is uncountable.

Let $A = \mathbf{Q} \cap [0,1]$. If $R_1,...,R_m$ are open intervals so that $A \subseteq \cup_{j=1}^{m} R_j$, prove that $1 \leq \sum_{j=1}^{m} \mathrm{vol}_1(R_j)$. Discuss the contrast to $m_1^*(A) = 0$.

Prove that the Cantor set is perfect: it is closed and has no isolated point.

The Cantor set and ternary expansions of numbers.

Prove that for every sequence (a_n) in {0,1,2} the series $\sum_{n=1}^{+\infty} \frac{a_n}{3^n}$ converges to a number in [0,1].

Conversely, prove that for every number x in [0,1] there is a sequence (a_n) in {0,1,2} so that $x = \sum_{n=1}^{+\infty} \frac{a_n}{3^n}$. Then we say that $0.a_1a_2$ ... is **a ternary expansion of** x and that $a_1, a_2, ...$ are **the ternary digits** of this expansion.

If $x \in [0,1]$ is a rational $\frac{m}{3^N}$, where $m \equiv 1 (\mathrm{mod} 3)$ and $N \in \mathbf{N}$, then x has exactly two ternary expansions: one is of the form $0.a_1 ... a_{N-1}1000...$ and the other is of the form $0.a_1 ... a_{N-1}0222...$.

If $x \in [0,1]$ is either irrational or rational $\frac{m}{3^N}$, where $m \equiv 0$ or $2 (mod 3)$ and $N \in \mathbf{N}$, then it has exactly one ternary expansion which is not of either one of the above forms.

Let C be the Cantor set. If $x \in [0,1]$, prove that $x \in C$ if and only if x has at least one ternary expansion containing *no* ternary digit 1.

The Cantor function.

Let $I_0 = [0,1], I_1, I_2, \ldots$ be the sets used in the construction of the Cantor set C. For each $n \in \mathbf{N}$ define $f_n : [0,1] \to [0,1]$ as follows. If, going from left to right, $J_1^{(n)}, \ldots, J_{2^n-1}^{(n)}$ are the $2^n - 1$ subintervals of $[0,1] \setminus I_n$, then define $f_n(0) = 0, f_n(1) = 1$, define f_n to be constant $\frac{k}{2^n}$ in $J_k^{(n)}$ for all $k = 1,\ldots,2^n-1$ and to be linear in each of the subintervals of I_n in such a way that f_n is continuous in $[0,1]$.

Prove that $|f_n(x) - f_{n-1}(x)| \leq \frac{1}{3 \cdot 2^n}$ for all $n \geq 2$ and all $x \in [0,1]$. This implies that for every $x \in [0,1]$ the series $f_1(x) + \sum_{k=2}^{+\infty} (f_k(x) - f_{k-1}(x))$ converges to a real number.

Define $f(x)$ to be the sum of the series appearing in (i) and prove that $|f(x) - f_n(x)| \leq \frac{1}{3 \cdot 2^n}$ for all $x \in [0,1]$. Therefore, f_n converges to f uniformly in $[0,1]$.

Prove that $f(0) = 0, f(1) = 1$ and that f is continuous and increasing in $[0,1]$.

Prove that for every n: f is constant $\frac{k}{2^n}$ in $J_k^{(n)}$ for all $k = 1,\ldots,2^n-1$. (v) Prove that, if $x,y \in C$ and $x < y$ and x,y are not end-points of the same complementary interval of C, then $f(x) < f(y)$.

This function f is called **the Cantor function**.

The difference set of a set.

Let $E \subseteq \mathbf{R}$ with $m_1^*(E) > 0$ and $0 \leq \alpha < 1$. Prove that there is a non-empty open interval (a,b) so that $m_1^*(E \cap (a,b)) \geq \alpha \cdot (b-a)$.

Let $E \subseteq \mathbf{R}$ be a Lebesgue set with $m_1(E) > 0$. Taking $\alpha = \frac{3}{4}$ in (i), prove that $E \cap (E + z) \cap (a,b)$ 6= ∅ for all z with $|z| < \frac{1}{4}(b-a)$.

Let $E \subseteq \mathbf{R}$ be a Lebesgue set with $m_1(E) > 0$. Prove that the set $D(E) = \{x - y \mid x,y \in E\}$, called **the difference set of** E, includes some open interval of the form $(-\epsilon, \epsilon)$.

Another construction of a non-Lebesgue set in **R**.

For any $x,y \in \mathbf{R}$ define $x \sim y$ if $x - y \in \mathbf{Q}$. Prove that $\sim$ is an equivalence relation in **R**.

Let L be a set containing exactly one element from each of the equivalence classes of $\sim$. Prove that $\mathbf{R} = \cup_{r\in\mathbf{Q}}(L + r)$ and that the sets $L + r$, $r \in \mathbf{Q}$, are pairwise disjoint.

Prove that the difference set of L (see exercise 4.6.11) contains no rational number 6= 0.

Using the result of exercise 4.6.11, prove that L is not a Lebesgue set.

Non-Lebesgue sets are everywhere, I.

We shall prove that every $E \subseteq \mathbf{R}$ with $m_1^*(E) >0$ includes at least one non-Lebesgue set.

Consider the non-Lebesgue set $N \subseteq [0,1]$ which was constructed in section 4.5 and prove that, if $B \subseteq N$ is a Lebesgue set, then $m_1(B) = 0$.

In other words, if $M \subseteq N$ has $m_1^*(M) >0$, then M is a non-Lebesgue set.

(ii) Consider an arbitrary $E \subseteq \mathbf{R}$ with $m_1^*(E) > 0$. If $\alpha = 1 - m_1^*(N)$, then $0 \le \alpha < 1$, and consider an interval (a,b) so that $m_1^*(E\cap(a,b)) \ge \alpha(b-a)$ (see exercise 4.6.11). Then the set $N^0 = (b - a)N + a$ is included in $[a,b]$, has $m_1^*(N') = (1-\alpha)\cdot(b-a)$ and, if $M^0 \subseteq N^0$ has $m_1^*(M') >0$, then M^0 is not a Lebesgue set.

(iii) Prove that $E \cap N^0$ is not a Lebesgue set.

No-Lebesgue sets are everywhere, II.

Consider the set L from exercise 4.6.12. Then $E = \cup_{r\in\mathbf{Q}}(E \cap (L + r))$ and prove that the difference set (exercise 4.6.11) of each $E \cap (L + r)$ contains no rational number 6= 0.

Prove that, for at least one $r \in \mathbf{Q}$, the set $E\cap(L+r)$ is not a Lebesgue set (using exercise 4.6.11).

Not all Lebesgue sets in **R** *are Borel sets and not all continuous functions map Lebesgue sets onto Lebesgue sets*.

Let $f : [0,1] \to [0,1]$ be the Cantor function constructed in exercise 4.6.10. Define $g : [0,1] \to [0,2]$ by the formula

$$g(x) = f(x) + x, \qquad x \in [0,1].$$

Prove that g is continuous, strictly increasing, one-to-one and onto [0,2]. Its inverse function $g^{-1} : [0,2] \to [0,1]$ is also continuous, strictly increasing, one-to-one and onto [0,1].

Prove that the set $g([0,1] \setminus C)$, where C is the Cantor set, is an open set with Lebesgue measure equal to 1. Therefore the set $E = g(C)$ has Lebesgue measure equal to 1.

Exercises 4.6.13 and 4.6.14 give non-Lebesgue sets $M \subseteq E$. Consider the set $K = g^{-1}(M) \subseteq C$. Prove that K is a Lebesgue set.

Using exercise 1.6.8, prove that K is not a Borel set in $\mathbf{R}$. (v) g maps K onto M.

More Cantor sets.

Take an arbitrary sequence $(_n)$ so that $0< \epsilon_n < \frac{1}{2}$for all n. We split $I_0 = [0,1]$ into the three intervals $[0, \frac{1}{2} - \epsilon_1], (\frac{1}{2} - \epsilon_1, \frac{1}{2} + \epsilon_1), [\frac{1}{2} + \epsilon_1, 1]$ and form I_1 as the union of the two closed intervals. Inductively, if we have already constructed I_{n-1} as a union of certain closed intervals, we split each of these intervals into three subintervals of which the two side ones are closed and their proportion to the original is $\frac{1}{2} - \epsilon_n$. The union of the new intervals is the I_n.

We set $K = \cap_{n=1}^{+\infty} I_n$.

Prove that K is compact, has no isolated points and includes no open interval.

Prove that K is uncountable.

Prove that $m_1(I_n) = (1 - 2\epsilon_1) \cdots (1 - 2\epsilon_n)$ for all n.

Prove that $m_1(K) = \lim_{n \to +\infty} (1 - 2\epsilon_1) \cdots (1 - 2\epsilon_n)$.

Taking $\epsilon_n = \frac{\epsilon}{3^n}$ for all n, prove that $m_1(K) > 1 - \epsilon$.

(Use that $(1 - a_1)\cdots(1 - a_n) > 1 - (a_1 + \cdots + a_n)$ for all n and all $a_1,...,a_n \in [0,1]$).

Prove that $m_1(K) > 0$ if and only if $\sum_{n=1}^{+\infty} \epsilon_n < +\infty$.

(Use the inequality you used for (v) and also that $1 - a \leq e^{-a}$ for all a.)

Uniqueness of Lebesgue measure.

Prove that m_n is the only measure μ on $(\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n})$ which is invariant under translations (i.e. $\mu(E + x) = \mu(E)$ for all Borel sets E and all x) and which satisfies $\mu(Q_0) = 1$, where $Q_0 = [-1,1] \times \cdots \times [-1,1]$.

Let $E \subseteq \mathbf{R}$ be a Lebesgue set and A be a dense subset of $\mathbf{R}$. If $m_1(E4(E+ x)) = 0$ for all $x \in A$, prove that $m_1(E) = 0$ or $m_1(E^c) = 0$.

Let $E \subseteq \mathbf{R}$ be a Lebesgue set and $\delta > 0$. If $m_1(E \cap (a,b)) \geq \delta(b - a)$ for all intervals (a,b), prove that $m_1(E^c) = 0$.

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4.12 ANSWERS TO CHECK YOUR PROGRESS

1.Please check section 4.3-4.7 for answers.

2.Please check section 4.3-4.7 for answers.

3.Please check section 4.3-4.7 for answers.

4.Please check section 4.3-4.7 for answers.

CHAPTER 5 BOREL MEASURES

STRUCTURE

5.1 OBJECTIVES

In this Chapter we are going to learn about the different kinds of borel measures

5.2 INTRODUCTION

The collection of μ^*_F-measurable sets is a σ-algebra of subsets of (a_0,b_0), which we denote by Σ_F, and the restriction, denoted μ_F, of μ^*_F on Σ_F is a complete measure.

The measure μ_F is called the Lebesgue-Stieltjes measure induced by the (increasing) $F : (a_0,b_0) \to \mathbf{R}$.

5.3 LEBESGUE-STIELTJES MEASURES IN R.

Lemma 5.1 *If $-\infty \le a < b \le +\infty$ and $F : (a,b) \to \mathbf{R}$ is increasing, then (i) for all $x \in [a,b)$ we have $F(x+) = \inf\{F(y)|x < y\}$,*

for all $x \in (a,b]$ we have $F(x-) = \sup\{F(y)|y < x\}$, *if* $a < x < y < z < b$, *then* $F(x-) \le F(x) \le F(x+) \le F(y) \le F(z-) \le F(z) \le F(z+)$,

for all $x \in [a,b)$ *we have* $F(x+) = \lim_{y\to x+} F(y\pm)$, *(v) for all* $x \in (a,b]$ *we have* $F(x-) = \lim_{y\to x-} F(y\pm)$.

Proof: (i) Let $M = \inf\{F(y)|x < y\}$. Then for every $\gamma > M$ there is some $t > x$ so that $F(t) < \gamma$. Hence for all $y \in (x,t)$ we have $M \le F(y) < \gamma$. This says that $F(x+) = M$.

Similarly, let $m = \sup\{F(y)|y < x\}$. Then for every $\gamma < m$ there is some $t < x$ so that $\gamma < F(t)$. Hence for all $y \in (t,x)$ we have $\gamma < F(y) \le m$. This says that $F(x-) = m$.

$F(x)$ is an upper bound of the set $\{F(y)|y < x\}$ and a lower bound of $\{F(y)|x < y\}$. This, by (i) and (ii), implies that $F(x-) \le F(x) \le F(x+)$ and, of course, $F(z-) \le F(z) \le F(z+)$. Also, if $x < y < z$, then $F(y)$ is an element of both sets $\{F(y)|x < y\}$ and $\{F(y)|y < z\}$. Therefore $F(y)$ is between the infimum of the first, $F(x+)$, and the supremum of the second set, $F(z-)$. (iv) By the result of (i), for every $\gamma > F(x+)$ there is some $t > x$ so that $F(x+) \le F(t) < \gamma$. This, combined with (iii), implies that $F(x+) \le F(y\pm) < \gamma$ for all $y \in (x,t)$. Thus, $F(x+) = \lim_{y\to x+} F(y\pm)$.

(v) By (ii), for every $\gamma < F(x-)$ there is some $t < x$ so that $\gamma < F(t) \le F(x-)$.

This, combined with (iii), implies $\gamma < F(y\pm) \le F(x-)$ for all $y \in (t,x)$. Thus, $F(x-) = \lim_{y\to x-} F(y\pm)$.

Consider now a_0,b_0 with $-\infty \le a_0 < b_0 \le +\infty$ and an increasing function $F : (a_0,b_0) \to \mathbf{R}$ and define a *non-negative* function τ acting on subintervals of (a_0,b_0) as follows:

$$\tau((a,b)) = F(b-) - F(a+), \qquad \tau([a,b]) = F(b+) - F(a-),$$
$$\tau((a,b]) = F(b+) - F(a+), \qquad \tau([a,b)) = F(b-) - F(a-).$$

The mnemonic rule is: if the end-point is included in the interval, then approach it from the outside while, if the end-point is not included in the interval, then approach it from the inside of the interval.

We use the collection of all open subintervals of (a_0,b_0) and the function τ to define, as an application of Theorem 3.2, the following outer measure on
(a_0,b_0):

o

$$\mu^*_F(E) = \inf\Big\{\sum_{j=1}^{+\infty}\tau((a_j,b_j))\,|\,E\subseteq\cup_{j=1}^{+\infty}(a_j,b_j),(a_j,b_j)\subseteq(a_0,b_0$$

for every $E \subseteq (a_0,b_0)$.

Theorem 3.1 implies that the collection of $\mu^*{}_F$-measurable sets is a σ-algebra of subsets of (a_0,b_0), which we denote by Σ_F, and the restriction, denoted μ_F, of $\mu^*{}_F$ on Σ_F is a complete measure.

Definition 5.1 *The measure μ_F is called the Lebesgue-Stieltjes measure induced by the (increasing) $F : (a_0,b_0) \to \mathbf{R}$.*

If $F(x) = x$ for all $x \in \mathbf{R}$, then $\tau(S) = \mathrm{vol}_1(S)$ for all intervals S and, in this special case, μ_F coincides with the 1-dimensional Lebesgue measure m_1 on $\mathbf{R}$. Thus, the new measure is a generalization of Lebesgue measure.

Following exactly the same procedure as with Lebesgue measure, we shall study the relation between the σ-algebra Σ_F and the Borel sets in (a_0,b_0).

Lemma 5.2 *Let $P = (a,b] \subseteq (a_0,b_0)$ and $a = c^{(0)} < c^{(1)} < \cdots < c^{(m)} = b$. If $P_i = (c^{(i-1)},c^{(i)}]$, then $\tau(P) = \tau(P_1) + \cdots + \tau(P_m)$.*

Proof: A telescoping sum:
$\tau(P_1)+\cdots+\tau(P_m) = \sum_{i=1}^{m}(F(c^{(i)}+)-F(c^{(i-1)}+)) =$
$F(b+) - F(a+) = \tau((a,b])$.

Lemma 5.3 *If $P,P_1,...,P_l$ are open-closed subintervals of (a_0,b_0), $P_1,...,P_l$ are pairwise disjoint and $P = P_1 \cup \cdots \cup P_l$, then $\tau(P) = \tau(P_1) + \cdots + \tau(P_l)$.*

Proof: Exactly one of $P_1,...,P_l$ has the same right end-point as P. We rename and call it P_l. Then exactly one of $P_1,...,P_{l-1}$ has right end-point coinciding with the left end-point of P_l. We rename and call it P_{l-1}. We continue until the left end-point of the last remaining subinterval, which we shall rename P_1, coincides with the left end-point of P. Then the result is the same as the result of Lemma 5.2.

Lemma 5.4 *If $P,P_1,...,P_l$ are open-closed subintervals of (a_0,b_0), $P_1,...,P_l$ are pairwise disjoint and $P_1 \cup \cdots \cup P_l \subseteq P$, then $\tau(P_1) + \cdots + \tau(P_l) \leq \tau(P)$.*

Proof: We know that $P \setminus (P_1 \cup \cdots \cup P_l) = P'_1 \cup \cdots \cup P'_k$ for some pairwise disjoint open-closed intervals $P'_1, \ldots, P'_k$. By Lemma 5.3 we get $\tau(P) = \tau(P_1) + \cdots +$

$\tau(P_l) + \tau(P'_1) + \cdots + \tau(P'_k) \geq \tau(P_1) + \cdots + \tau(P_l)$.

Lemma 5.5 *Suppose that $P,P_1,...,P_l$ are open-closed subintervals of (a_0,b_0) and $P \subseteq P_1 \cup \cdots \cup P_l$. Then $\tau(P) \leq \tau(P_1) + \cdots + \tau(P_l)$.*

Proof: We write $P = P'_1 \cup \cdots \cup P'_l$, where $P_j^0 = P_j \cap P$ are open-closed intervals included in P. Then write $P = P'_1 \cup (P'_2 \setminus P'_1) \cup \cdots \cup \left(P'_l \setminus (P'_1 \cup \cdots \cup P'_{l-1})\right)$.

Each of these l pairwise disjoint sets can be written as a finite union of pairwise disjoint open-closed intervals: $P'_1 = P'_1$ and

$P'_j \setminus (P'_1 \cup \cdots \cup P'_{j-1}) = P_1^{(j)} \cup \cdots \cup P_{m_j}^{(j)}$

for $2 \leq j \leq l$.

Lemma 5.3 (for the equality) and Lemma 5.4 (for the two inequalities) imply

$$\begin{aligned} \tau(P) &= \tau(P'_1) + \sum_{j=2}^{l} \Big(\sum_{m=1}^{m_j} \tau(P_m^{(j)}) \Big) \\ &\leq \tau(P'_1) + \sum_{j=2}^{l} \tau(P'_j) \leq \sum_{j=1}^{l} \tau(P_j) \end{aligned}.$$

Lemma 5.6 *Let Q be a closed interval and $R_1,...,R_l$ be open subintervals of (a_0,b_0). If $Q \subseteq R_1 \cup \cdots \cup R_l$, then $\tau(Q) \leq \tau(R_1) + \cdots + \tau(R_l)$.*

Proof: Let $Q = [a,b]$ and $R_j = (a_j,b_j)$ for $j = 1,...,l$. We define for $\epsilon > 0$

$$P_\epsilon = (a - \epsilon, b], \qquad P_{j,\epsilon} = (a_j, b_j - \epsilon].$$

We shall first prove that there is some $\epsilon_0 > 0$ so that for all $\epsilon < \epsilon_0$

$$P_\epsilon \subseteq P_{1,\epsilon} \cup \cdots \cup P_{l,\epsilon}.$$

Suppose that, for all n, the above inclusion is not true for $\epsilon = \frac{1}{n}$. Hence, for all n there is $x_n \in (a - \frac{1}{n}, b]$ so that $x_n \notin \cup_{j=1}^{l} (a_j, b_j - \frac{1}{n}]$. By the Bolzano-Weierstrass theorem, there is a subsequence (x_{nk}) converging to some x. Looking carefully at the various inequalities, we get $x \in [a,b]$ and $x \notin$

$\cup^{l}_{j=1}(a_j,b_j)$. This is a contradiction and the inclusion we want to prove is true for some $\epsilon_0 = \frac{1}{n_0}$. If

$\epsilon < \epsilon_0$, then the inclusion is still true because the left side becomes smaller while the right side becomes larger.

Now Lemma 5.5 gives for $\epsilon < \epsilon_0$ that

$$F(b+) - F((a-\epsilon)+) \le \sum_{j=1}^{l} \big(F((b_j-\epsilon)+) - F(a_j+)\big)$$

and, using Lemma 5.1 for the limit as $\epsilon \to 0+$,

$$\tau(Q) = F(b+) - F(a-) \le \sum_{j=1}^{l} \big(F(b_j-) - F(a_j+)\big) = \sum_{j=1}^{l} \tau(R_j)$$

Theorem 5.1 *Let $F : (a_0,b_0) \to \mathbf{R}$ be increasing. Then every subinterval S of (a_0,b_0) is μ^*_F-measurable and*

$$\mu_F(S) = \tau(S).$$

Proof: Let $Q = [a,b] \subseteq (a_0,b_0)$.

Then $\mu^*_F(Q) \le \tau((a-\epsilon, b+\epsilon)) = F((b+\epsilon)-) - F((a-\epsilon)+)$ for all small enough > 0 and, thus, $\mu^*_F(Q) \le F(b+) - F(a-) = \tau(Q)$.

For every covering $Q \subseteq \cup_{j=1}^{+\infty} R_j$ by open subintervals of (a_0,b_0), there is (by compactness) l so that $Q \subseteq \cup^{l}_{j=1} R_j$. Lemma 5.6 implies $\tau(Q) \le \sum_{j=1}^{l} \tau(R_j) \le$

$\sum_{j=1}^{+\infty} \tau(R_j)$. Hence $\tau(Q) \le \mu^*_F(Q)$ and we conclude that

$\tau(Q) = \mu^*_F(Q)$

for all closed intervals $Q \subseteq (a_0,b_0)$.

If $P = (a,b] \subseteq (a_0,b_0)$, then $\mu^*_F(P) \le \tau((a, b+\epsilon)) = F((b+\epsilon)-) - F(a+)$ for all small enough > 0. Hence $\mu^*_F(P) \le F(b+) - F(a+) = \tau(P)$.

If $R = (a,b) \subseteq (a_0,b_0)$, then $\mu^*_F(R) \le \tau((a,b)) = \tau(R)$.

Now let $P = (a,b]$, $R = (c,d)$ be included in (a_0,b_0) and take $P_R = (c, d-\epsilon]$. We write $\mu^*_F(R \cap P) = \mu^*_F\big((P_R \cap P) \cup ((d-\epsilon, d) \cap P)\big) \le \mu^*_F(P_R \cap P) +$

$\mu^*_F((d-\epsilon, d)) \le \tau(P_R \cap P) + F(d-) - F((d-\epsilon)+)$ by the previous results. The same inequalities, with P^c instead of P, give $\mu^*_F(R \cap P^c) \le \mu^*_F(P_R \cap P^c) + F(d-) - F((d-\epsilon)+)$. Taking the sum, we find $\mu^*_F(R \cap P) + \mu^*_F(R \cap P^c) \le$

$\tau(P_R \cap P) + \mu^*_F(P_R \cap P^c) + 2[F(d-) - F((d-\epsilon)+)]$.

Now write $P_R \cap P^c = P_1 \cup \cdots \cup P_l$ for pairwise disjoint open-closed intervals and get $\tau(P_R \cap P) + \mu_F^*(P_R \cap P^c) \leq \tau(P_R \cap P) + \sum_{j=1}^{l} \mu_F^*(P_j) \leq \tau(P_R \cap P) + \sum_{j=1}^{l} \tau(P_j) = \tau(P_R)$ by the first results and Lemma 5.3.

Therefore $\mu_F^*(R \cap P) + \mu_F^*(R \cap P^c) \leq \tau(P_R) + 2[F(d-) - F((d-\epsilon)+)] = F((d-\epsilon)+) - F(c+) + 2[F(d-) - F((d-\epsilon)+)]$ and, taking limit, $\mu_F^*(R \cap P) + \mu_F^*(R \cap P^c) \leq F(d-) - F(c+) = \tau(R)$.

We proved that

$\mu_F^*(R \cap P) + \mu_F^*(R \cap P^c) \leq \tau(R)$

for all open intervals R and open-closed intervals P which are $\subseteq (a_0,b_0)$.

Now consider arbitrary $E \subseteq (a_0,b_0)$ with $\mu_F^*(E) < +\infty$. Take a covering $E \subseteq \cup_{j=1}^{+\infty} R_j$ by open subintervals of (a_0,b_0) so that $\sum_{j=1}^{+\infty} \tau(R_j) < \mu_F^*(E) + \epsilon$.

By σ-subadditivity and the last result we find $\mu_F^*(E \cap P) + \mu_F^*(E \cap P^c) \leq \sum_{j=1}^{+\infty} \left(\mu_F^*(R_j \cap P) + \mu_F^*(R_j \cap P^c)\right) \leq \sum_{j=1}^{+\infty} \tau(R_j) < \mu_F^*(E) + \epsilon$.

Taking limit as $\epsilon \to 0+$, we find

$$\mu_F^*(E \cap P) + \mu_F^*(E \cap P^c) \leq \mu_F^*(E),$$

concluding that $P \in \Sigma_F$.

If $Q = [a,b] \subseteq (a_0,b_0)$, we take any (a_k) in (a_0,b_0) so that $a_k \uparrow a$ and, then, $Q = \cap_{k=1}^{+\infty}(a_k, b] \in \Sigma_F$. Moreover, by the first results,

$$\mu_F(Q) = \mu_F^*(Q) = \tau(Q).$$

If $P = (a,b] \subseteq (a_0,b_0)$, we take any (a_k) in $(a,b]$ so that $a_k \downarrow a$ and we get that $\mu_F(P) = \lim_{k\to+\infty} \mu_F([a_k,b]) = \lim_{k\to+\infty}(F(b+) - F(a_k-)) = F(b+) - F(a+) = \tau(P)$.

If $T = [a,b) \subseteq (a_0,b_0)$, we take any (b_k) in $[a,b)$ so that $b_k \uparrow b$ and we get that $T = \cup_{k=1}^{+\infty}[a, b_k] \in \Sigma_F$. Moreover, $\mu_F(T) = \lim_{k\to+\infty} \mu_F([a,b_k]) = \lim_{k\to+\infty}(F(b_k+) - F(a-)) = F(b-) - F(a-) = \tau(T)$.

Finally, if $R = (a,b) \subseteq (a_0,b_0)$, we take any (a_k) and (b_k) in (a,b) so that $a_k \downarrow a$, $b_k \uparrow b$ and $a_1 \leq b_1$. Then $R = \cup_{k=1}^{+\infty}[a_k, b_k] \in \Sigma_F$. Moreover, $\mu_F(R) = \lim_{k\to+\infty} \mu_F([a_k,b_k]) = \lim_{k\to+\infty}(F(b_k+) - F(a_k-)) = F(b-) - F(a+) = \tau(R)$.

Theorem 5.2 *Let $F : (a_0,b_0) \to$ **R** be increasing. Then μ_F is σ-finite and it is finite if and only if F is bounded. Also,* $\mu_F((a_0, b_0)) = F(b_0-) - F(a_0+)$.

Proof: We consider any two sequences (a_k) and (b_k) in (a_0,b_0) so that $a_k \downarrow a_0$, $b_k \uparrow b_0$ and $a_1 \leq b_1$. Then $(a_0, b_0) = \cup_{k=1}^{+\infty}[a_k, b_k]$ and $\mu_F([a_k,b_k]) = F(b_k+) -$

$F(a_k-) < +\infty$ for all k. Hence, μ_F is σ-finite.

Since $\mu_F((a_0,b_0)) = F(b_0-) - F(a_0+)$, if μ_F is finite, then $-\infty < F(a_0+)$ and $F(b_0-) < +\infty$. This implies that all values of F lie in the bounded interval $[F(a_0+),F(b_0-)]$ and F is bounded. Conversely, if F is bounded, then the limits $F(a_0+),F(b_0-)$ are finite and $\mu_F((a_0,b_0)) < +\infty$.

It is easy to prove that *the collection of all subintervals of* (a_0,b_0) *generates the* σ*-algebra of all Borel sets in* (a_0,b_0). Indeed, let E be the collection of all intervals in **R** and F be the collection of all subintervals of (a_0,b_0). It is clear that F = E$\rceil(a_0,b_0)$ and Theorems 1.2 and 1.3 imply that

$$B_{(a_0,b_0)} = B_{\mathbf{R}}\rceil(a_0,b_0) = \Sigma(E)\rceil(a_0,b_0) = \Sigma(F).$$

Theorem 5.3 *Let* $F : (a_0,b_0) \to \mathbf{R}$ *be increasing. Then all Borel sets in* (a_0,b_0) *belong to* Σ_F.

Proof: Theorem 5.1 implies that the collection F of all subintervals of (a_0,b_0) is included in Σ_F. By the discussion of the previous paragraph, we conclude that $B(a0,b0) = \Sigma(F) \subseteq \Sigma F$.

Theorem 5.4 *Let* $F : (a_0,b_0) \to \mathbf{R}$ *be increasing. Then for every* $E \subseteq (a_0,b_0)$ *we have*

$E \in \Sigma_F$ *if and only if there is* $A \subseteq (a_0,b_0)$, *a countable intersection of open sets, so that* $E \subseteq A$ *and* $\mu^*_F(A \setminus E) = 0$.

$E \in \Sigma_F$ *if and only if there B, a countable union of compact sets, so that B* $\subseteq E$ *and* $\mu^*_F(E \setminus B) = 0$.

Proof: The proof is exactly the same as the proof of the similar Theorem 4.4. Only the obvious changes have to be made: m_n changes to μ_F and m^*_n to μ^*_F, $\mathbf{R}^n$ changes to (a_0,b_0), vol_n changes to τ and L_n changes to Σ_F.

Therefore, *every set in* Σ_F *is, except from a* μ_F*-null set, equal to a Borel set*.

Theorem 5.5 *Let* $F : (a_0,b_0) \to \mathbf{R}$ *be increasing. Then* *(i)* μ_F *is the only measure on* $(a_0, b_0), \mathcal{B}_{(a_0,b_0)})$ *with* $\mu_F((a,b]) = F(b+) - F(a+)$ *for all intervals* $(a,b] \subseteq (a_0,b_0)$. *(ii)* $((a_0, b_0), \Sigma_F, \mu_F)$ *is the completion of* $(a_0, b_0), \mathcal{B}_{(a_0,b_0)}, \mu_F)$.

Proof: The proof is similar to the proof of Theorem 4.5. Only the obvious notational modifications are needed.

It should be observed that the measure of a set $\{x\}$ consisting of a single point $x \in (a_0,b_0)$ is equal to $\mu_F(\{x\}) = F(x+) - F(x-)$, the jump of F at x. In other words, the measure of a one-point set is positive if and only if F is discontinuous there. Also, observe that the measure of an *open* subinterval of (a_0,b_0) is 0 if and only if F is constant in this interval.

It is very common in practice to consider the increasing function F with the extra property of being *continuous from the right*. In this case the measure of an open-closed interval takes the simpler form

$$\mu_F((a,b]) = F(b) - F(a).$$

Proposition 5.1 shows that this is not a serious restriction.

Proposition 5.1 *Given any increasing function on* (a_0,b_0) *there is another increasing function which is continuous from the right so that the LebesgueStieltjes measures induced by the two functions are equal.*

Proof: Given any increasing $F : (a_0,b_0) \to \mathbf{R}$ we define $F_0 : (a_0,b_0) \to \mathbf{R}$ by the formula

$$F_0(x) = F(x+), \qquad x \in (a_0,b_0)$$

and it is immediate from Lemma 5.1 that F_0 is increasing, continuous from the right, i.e. $F_0(x+) = F_0(x)$ for all x, and $F_0(x+) = F(x+), F_0(x-) = F(x-)$ for all x. Now, it is obvious that F_0 and F induce the same Lebesgue-Stieltjes measure on (a_0,b_0), simply because the corresponding functions $\tau(S)$ (from which the construction of the measures μ_{F0}, μ_F starts) assign the same values to every interval $S \subseteq (a_0,b_0)$.

The functions F_0 and F of Proposition 5.1 have the same jump at every x and, in particular, they have the same continuity points.

Check your progress

1.*Supports of Lebesgue-Stieltjes measures.*

Let $F : \mathbf{R} \to \mathbf{R}$ be any increasing function. Prove that the complement of the support (exercise 5.5.6) of the measure μ_F is the union of all *open* intervals on each of which F is constant.

Let $a : \mathbf{R} \to [0,+\infty]$ induce the point-mass distribution μ on $(\mathbf{R},\mathcal{P}(\mathbf{R}))$. Then μ is a Borel measure on $\mathbf{R}$.

Prove that μ is locally finite if and only if $\sum_{-R\leq x\leq R} a_x < +\infty$ for all $R > 0$.

In particular, prove that, if μ is locally finite, then $A = \{x \in \mathbf{R} | a_x > 0\}$ is countable.

In case μ is locally finite, find an increasing, continuous from the right $F : \mathbf{R} \to \mathbf{R}$ (in terms of the function a) so that $\mu = \mu_F$ on $\mathcal{B}_{\mathbf{R}}$.

Describe the sets E such that $\mu_F^*(E) = 0$ and find the σ-algebra Σ_F of all μ_F^*-measurable sets. Is $\Sigma_F = \mathcal{P}(\mathbf{R})$?

5.4 BOREL MEASURES.

Definition 5.2 *Let X be a topological space and (X,Σ,μ) be a measure space. The measure μ is called a Borel measure on X if $\mathcal{B}_X \subseteq \Sigma$, i.e. if all Borel sets in X are in Σ.*

The Borel measure μ is called locally finite if for every $x \in X$ there is some open neighborhood U_x of x (i.e. an open set containing x) such that $\mu(U_x) < +\infty$.

Observe that, for μ to be a Borel measure, it is enough to have that all open sets or all closed sets are in Σ. This is because $\mathcal{B}_X$ is generated by the collections of all open or all closed sets and because Σ is a σ-algebra.

Examples

The Lebesgue measure on $\mathbf{R}^n$ and, more generally, the Lebesgue-Stieltjes measure on any generalized interval (a_0,b_0) (induced by any increasing function) are locally finite Borel measures. In fact, the content of the following theorem is that the only locally finite Borel measures on (a_0,b_0) are exactly the Lebesgue-Stieltjes measures.

Lemma 5.7 *Let X be a topological space and μ a Borel measure on X. If μ is locally finite, then $\mu(K) < +\infty$ for every compact $K \subseteq X$.*

If μ is a locally finite Borel measure on $\mathbf{R}^n$*, then* $\mu(M) < +\infty$ *for every bounded* $M \subseteq \mathbf{R}^n$*.*

Proof: We take for each $x \in K$ an open neighborhood U_x of x so that $\mu(U_x) <$

$+\infty$.Since $K \subseteq \cup_{x\in K} U_x$ and K is compact, there are $x_1,...,x_n$ so that

$K \subseteq \cup_{k=1}^{n} U_{x_k}$. Hence, $\mu(K) \leq \sum_{k=1}^{n} \mu(U_{x_k}) < +\infty$.

If $M \subseteq \mathbf{R}^n$ is bounded, then M is compact and $\mu(M) \leq \mu(M) < +\infty$.

Theorem 5.6 *Let* $-\infty \leq a_0 < b_0 \leq +\infty$ *and* $c_0 \in (a_0,b_0)$*. For every locally finite Borel measure* μ *on* (a_0,b_0) *there is a unique increasing and continuous from the right* $F : (a_0,b_0) \to \mathbf{R}$ *so that* $\mu = \mu_F$ *on* $\mathrm{B}_{(a0,b0)}$ *and* $F(c_0) = 0$*. For any other increasing and continuous from the right* $G : (a_0,b_0) \to \mathbf{R}$*, it is true that* $\mu = \mu_G$ *if and only if* G *differs from* F *by a constant.*

Proof: Define the function

,

$$F(x) = \begin{cases} \mu((c_0,x]), & \text{if } c_0 \leq x < b_0 \\ -\mu((x,c_0]), & \text{if } a_0 < x < c \end{cases}$$

0.

By Lemma 5.7, F is real valued and it is clear, by the monotonicity of μ, that F is increasing. Now take any decreasing sequence (x_n) so that $x_n \downarrow x$. If $c_0 \leq x$, by continuity of μ from above, $\lim_{n\to+\infty} F(x_n) = \lim_{n\to+\infty} \mu((c_0,x_n]) = \mu((c_0,x]) = F(x)$. Also, if $x < c_0$, then $x_n < c_0$ for large n, and, by continuity of μ from below, $\lim_{n\to+\infty} F(x_n) = -\lim_{n\to+\infty} \mu((x_n,c_0]) = -\mu((x,c_0]) = F(x)$. Therefore, F is continuous from the right at every x.

If we compare μ and the induced μ_F at the intervals $(a,b]$, we get $\mu_F((a,b]) = F(b)-F(a) = \mu((a,b])$, where the second equality becomes trivial by considering cases: $a < b < c_0$, $a < c_0 \leq b$ and $c_0 \leq a < b$. Theorem 5.5 implies that $\mu_F = \mu$

on $\mathrm{B}(a_0,b_0)$.

If G is increasing, continuous from the right with $\mu_G = \mu (= \mu_F)$ on $\mathrm{B}_{(a0,b0)}$, then $G(x) - G(c_0) = \mu_G((c_0,x]) = \mu_F((c_0,x]) = F(x) - F(c_0)$ for all $x \geq c_0$ and, similarly, $G(c_0) - G(x) = \mu_G((x,c_0]) = \mu_F((x,c_0]) = F(c_0) - F(x)$ for all $x < c_0$. Therefore F,G differ by a constant: $G-F = G(c_0)-F(c_0)$ on (a_0,b_0). Hence, if $F(c_0) = 0 = G(c_0)$, then F,G are equal on (a_0,b_0).

If the locally finite Borel measure μ on (a_0,b_0) satisfies the $\mu((a_0,c_0]) < +\infty$, then we may make a different choice for F than the one in Theorem 5.6. We add the constant $\mu((a_0,c_0])$ to the function of the theorem and get the function

$$F(x) = \mu((a_0,x]), \qquad x \in (a_0,b_0).$$

This last function is called **the cumulative distribution function of** μ.

A central notion related to Borel measures is the notion of regularity, and this is because of the need to replace the general Borel set (a somewhat obscure object) by open or closed sets.

Let E be a Borel subset in a topological space X and μ a Borel measure on X. It is clear that $\mu(K) \le \mu(E) \le \mu(U)$ for all K compact and U open with

$K \subseteq E \subseteq U$. Hence $\sup\{\mu(K) | K \text{ compact} \subseteq E\} \le \mu(E) \le \inf\{\mu(U) | U \text{ open} \supseteq E\}$.

Definition 5.3 *Let X be a topological space and μ a Borel measure on X. Then μ is called regular if the following are true for every Borel set E in X: (i) $\mu(E) = \inf\{\mu(U) | U \text{ open} \supseteq E\}$, (ii) $\mu(E) = \sup\{\mu(K) | K \text{ compact} \subseteq E\}$.*

Therefore, μ is regular if the measure of every Borel set can be approximated from above by the measures of larger open sets and from below by the measures of smaller compact sets.

Proposition 5.2 *Let O be any open set in $\mathbf{R}^n$. There is an increasing sequence (K_m) of compact subsets of O so that $\mathrm{int}(K_m) \uparrow O$ and, hence, $K_m \uparrow O$ also.*

Proof: Define the sets

no $\qquad |y-x| \ge \frac{1}{m}$

$K_m =$ $\quad x \in O \,\big|\, |x| \le m$ andfor all $y \notin O$,

where $|x|^2 = x_1^2 + \cdots + x_n^2$ for all $x = (x_1,\dots,x_n)$.

The set K_m is bounded, since $|x| \le m$ for all $x \in K_m$.

If (x_j) is a sequence in K_m converging to some x, then, from $|x_j| \le m$ for all j, we get $|x| \le m$, and, from $|y-x_j| \ge \frac{1}{m}$ for all j and for all $y \notin O$, we get $|y-x| \ge \frac{1}{m}$ for all $y \notin O$. Thus, $x \in K_m$ and K_m is closed.

Therefore, K_m is a compact subset of O and, clearly, $K_m \subseteq K_{m+1} \subseteq O$ for all m. Hence, $\mathrm{int}(K_m) \subseteq \mathrm{int}(K_{m+1})$ for every m.

Now take any $x \in O$ and an $\epsilon > 0$ such that $B(x;2\epsilon) \subseteq O$. Consider, also, $M \geq \max(|x| + \epsilon, \frac{1}{\epsilon})$. It is trivial to see that $B(x;\epsilon) \subseteq K_M$ and thus $x \in \text{int}(K_M)$. Therefore, $\text{int}(K_m) \uparrow O$. Since $\text{int}(K_m) \subseteq K_m \subseteq O$, we conclude that $K_m \uparrow O$.

Theorem 5.7 *Let X be a topological space with the property that for every open set O in X there is an increasing sequence of compact subsets of O whose interiors cover O.*

Suppose that μ is a locally finite Borel measure on X. Then:

For every Borel set E and every $\epsilon > 0$ *there is an open U and a closed F so that* $F \subseteq E \subseteq U$ *and* $\mu(U \setminus E), \mu(E \setminus F) < \epsilon$. *If also* $\mu(E) < +\infty$, *then F can be taken compact.*

For every Borel set E in X there is A, a countable intersection of open sets, and B, a countable union of compact sets, so that $B \subseteq E \subseteq A$ *and* $\mu(A \setminus E) = \mu(E \setminus B) = 0$.

μ is regular.

Proof: (a) Suppose that $\mu(X) < +\infty$.

Consider the collection S of all Borel sets E in X with the property expressed in (i), namely, that for every $\epsilon > 0$ there is an open U and a closed F so that

$F \subseteq E \subseteq U$ and $\mu(U \setminus E), \mu(E \setminus F) < \epsilon$.

Take any open set $O \subseteq X$ and arbitrary $\epsilon > 0$. If we consider $U = O$, then $\mu(U \setminus O) = 0 < \epsilon$. By assumption there is a sequence (K_m) of compact sets so that $K_m \uparrow O$. Therefore, $O \setminus K_m \downarrow \emptyset$ and, since $\mu(O \setminus K_1) \leq \mu(X) < +\infty$, continuity from above implies that $\lim_{m\to+\infty} \mu(O \setminus K_m) = 0$. Therefore there is some m so that $\mu(O \setminus F) < \epsilon$, if $F = K_m$.

Thus, all open sets belong to S.

If $E \in \mathcal{S}$ and $\epsilon > 0$ is arbitrary, we find an open U and a closed F so that $F \subseteq E \subseteq U$ and $\mu(U \setminus E), \mu(E \setminus F) < \epsilon$. Then F^c is open, U^c is closed, $U^c \subseteq E^c \subseteq F^c$ and $\mu(F^c \setminus E^c) = \mu(E \setminus F) < \epsilon$ and $\mu(E^c \setminus U^c) = \mu(U \setminus E) < \epsilon$.

This implies that $E^c \in$ S.

Now, take $E_1, E_2, \ldots \in$ S and $E = \cup_{j=1}^{+\infty} E_j$. For $\epsilon > 0$ and each E_j take open U_j and closed F_j so that $F_j \subseteq E_j \subseteq U_j$ and $\mu(U_j \setminus E_j), \mu(E_j \setminus F_j) < \frac{\epsilon}{2^j}$. Define $B = \cup_{j=1}^{+\infty} F_j$ and the open $U = \cup_{j=1}^{+\infty} U_j$ so that $B \subseteq E \subseteq U$. Then

$U \setminus E \subseteq \cup_{j=1}^{+\infty}(U_j \setminus E_j)$ and $E \setminus B \subseteq \cup_{j=1}^{+\infty}(E_j \setminus F_j)$. This implies $\mu(U \setminus E) \leq \sum_{j=1}^{+\infty} \mu(U_j \setminus E_j) < \sum_{j=1}^{+\infty} \frac{\epsilon}{2^j} = \epsilon$ and, similarly, $\mu(E \setminus B) < \epsilon$. The problem now

is that B is not necessarily closed. Consider the closed sets $F_j^0 = F_1 \cup \cdots \cup F_j$, so that $F_j^0 \uparrow B$. Then $E \setminus F_j^0 \downarrow E \setminus B$ and, since $\mu(E \setminus F_1') \leq \mu(X) < +\infty$, continuity from below implies $\mu(E \setminus F_j') \downarrow \mu(E \setminus B)$. Therefore there is some j so that $\mu(E \setminus F_j') < \epsilon$. The inclusion $F_j' \subseteq E$ is clearly true.

We conclude that $E = \cup_{j=1}^{+\infty} E_j \in \mathcal{S}$ and S is a σ-algebra.

Since S contains all open sets, we have that $B_X \subseteq$ S and finish the proof of the first statement of (i) in the special case $\mu(X) < +\infty$.

(b) Now, consider the general case, and take any Borel set E in X which is included in some compact set $K \subseteq X$. For each $x \in K$ we take an open neighborhood U_x of x with $\mu(U_x) < +\infty$. By the compactness of K, there exist $x_1,...,x_n \in K$ so that $K \subseteq \cup_{k=1}^{n} U_{x_k}$. We form the open set $G = \cup^n_{k=1} U_{xk}$ and have that

$$E \subseteq G, \qquad \mu(G) < +\infty.$$

We next consider the restriction μ_G of μ on G, which is defined by the formula

$$\mu_G(A) = \mu(A \cap G)$$

for all Borel sets A in X. It is clear that μ_G is a Borel measure on X which is finite, since $\mu_G(X) = \mu(G) < +\infty$.

By (a), for every > 0 there is an open U and a closed F so that $F \subseteq E \subseteq U$ and $\mu_G(U \setminus E), \mu_G(E \setminus F) < \epsilon$. Since $E \subseteq G$, we get $\mu((G \cap U) \setminus E) =$ $\mu(G \cap (U \setminus E)) = \mu_G(U \setminus E) < \epsilon$ and $\mu(E \setminus F) = \mu(G \cap (E \setminus F)) = \mu_G(E \setminus F) < \epsilon$.

Therefore, if we consider the open set $U^0 = G \cap U$, we get $F \subseteq E \subseteq U^0$ and $\mu(U' \setminus E), \mu(E \setminus F) < \epsilon$ and the first statement of (i) is now proved with no

restriction on $\mu(X)$ but only for Borel sets in X which are included in compact subsets of X.

(c) We take an increasing sequence (K_m) of compact sets so that $\text{int}(K_m) \uparrow X$.

For any Borel set E in X we consider the sets $E_1 = E \cap K_1$ and $E_m = E \cap (K_m \setminus K_{m-1})$ for all $m \geq 2$ and we have that $E = \cup_{m=1}^{+\infty} E_m$. Since $E_m \subseteq K_m$, (b) implies that for each m and every > 0 there is an open U_m and a closed

F_m so that $F_m \subseteq E_m \subseteq U_m$ and $\mu(U_m \setminus E_m), \mu(E_m \setminus F_m) < \frac{\epsilon}{2^m}$. Now define the open $U = \cup_{m=1}^{+\infty} U_m$ and the closed (why?) $F = \cup_{m=1}^{+\infty} F_m$, so that $F \subseteq E \subseteq U$. As in the proof of (a), we easily get $\mu(U \setminus E), \mu(E \setminus F) < \epsilon$.

This concludes the proof of the first statement of (i).

Let $\mu(E) < +\infty$. Take a closed F so that $F \subseteq E$ and $\mu(E \setminus F) < \epsilon$, and consider the compact sets K_m of part (c). Then the sets $F_m = F \cap K_m$ are compact and $F_m \uparrow F$. Therefore, $E \setminus F_m \downarrow E \setminus F$ and, by continuity of μ from above, $\mu(E \setminus F_m) \to \mu(E \setminus F)$. Thus there is a large enough m so that $\mu(E \setminus F_m) < \epsilon$. This proves the second statement of (i).

Take open U_j and closed F_j so that $F_j \subseteq E \subseteq U_j$ and $\mu(U_j \setminus E), \mu(E \setminus F_j) < \frac{1}{j}$. Define $A = \cap_{j=1}^{+\infty} U_j$ and $B = \cup_{j=1}^{+\infty} F_j$ so that $B \subseteq E \subseteq A$. Now, for all j we have $\mu(A \setminus E) \leq \mu(U_j \setminus E) < \frac{1}{j}$ and $\mu(E \setminus B) \leq \mu(E \setminus F_j) < \frac{1}{j}$. Therefore, $\mu(A \setminus E) = \mu(E \setminus B) = 0$. We define the compact sets $K_{j,m} = F_j \cap K_m$ and observe that $B = \cup_{(j,m) \in \mathbf{N} \times \mathbf{N}} K_{j,m}$. This is the proof of (ii).

If $\mu(E) = +\infty$, it is clear that $\mu(E) = \inf\{\mu(U) | U \text{ open and } E \subseteq U\}$. Also, from (ii), there is some $B = \cup_{m=1}^{+\infty} K'_m$, where all K'_m are compact, so that $B \subseteq E$ and $\mu(B) = \mu(E) = +\infty$. Consider the compact sets $K_m = K'_1 \cup \cdots \cup K'_m$ which satisfy $K_m \uparrow B$. Then $\mu(K_m) \to \mu(B) = \mu(E)$ and thus $\sup\{\mu(K) | K \text{ compact and } K \subseteq E\} = \mu(E)$.

If $\mu(E) < +\infty$, then, from (a), for every > 0 there is a compact K and an open U so that $K \subseteq E \subseteq U$ and $\mu(U \setminus E), \mu(E \setminus K) < \epsilon$. This implies $\mu(E) - \epsilon < \mu(K)$ and $\mu(U) < \mu(E) + \epsilon$ and, thus, the proof of (iii) is complete.

Lemma 5.8 *Let X be a topological space which satisfies the assumptions of Theorem 5.7. Let Y be an open or a closed subset of X with its subspace topology. Then Y also satisfies the assumptions of Theorem 5.7.*

Proof Let Y be open in X. If O is an open subset of Y, then it is also an open subset of X. Therefore, there is an increasing sequence (K_m) of compact subsets of O so that $\mathrm{int}_X(K_m) \uparrow O$, where $\mathrm{int}_X(K_m)$ is the interior of K_m with respect to X. Since $K_m \subseteq Y$ and Y is open in X, it is clear that $\mathrm{int}_Y(K_m) = \mathrm{int}_X(K_m)$ and, thus, $\mathrm{int}_Y(K_m) \uparrow O$.

Let Y be closed in X and take any $O \subseteq Y$ which is open in Y. Then $O = O^0 \cap Y$ for some $O^0 \subseteq X$ which is open in X and, hence, there is an increasing sequence (K'_m) of compact subsets of O^0 so that $\mathrm{int}_X(K'_m) \uparrow O'$. We set $K_m = K'_m \cap Y$ and have that each K_m is a compact subset of O. Moreover, $\mathrm{int}_X(K'_m) \cap Y \subseteq \mathrm{int}_Y(K_m)$ for every m and, thus, $\mathrm{int}_Y(K_m) \uparrow O$.

Examples

Proposition 5.2 implies that the euclidean space $\mathbf{R}^n$ satisfies the assumptions of Theorem 5.7. Therefore, *every locally finite Borel measure on* $\mathbf{R}^n$ *is regular*.

A special case of this is the Lebesgue measure in $\mathbf{R}^n$ (see Theorem 4.4 and

Exercice 4.6.5).

If Y is an *open* or a *closed* subset of $\mathbf{R}^n$ with the subspace topology, then Lemma 5.8 together with Theorem 5.7 imply that *every locally finite Borel measure on Y is regular*.

As a special case, if $Y = (a_0,b_0)$ is a generalized interval in $\mathbf{R}$, then every locally finite Borel measure on Y is regular. Since Theorem 5.6 says that any such measure is a Lebesgue-Stieltjes measure, this result is, also, easily implied by Theorem 5.4.

Check your progress

2.Linear combinations of regular Borel measures.

If μ,μ_1,μ_2 are regular Borel measures on the topological space X and $\lambda \in [0,+\infty)$, prove that $\lambda\mu$ and $\mu_1 + \mu_2$ are regular Borel measures on X.

Prove that every locally finite Borel measure on $\mathbf{R}^n$ is σ-finite.

5.5 LET US SUM UP

In this unit we discussed the following

Lebesgue-Stieltjes measures in **R**

Borel measures

5.6 KEYWORDS

Borel measures-
In mathematics, specifically in measure theory, a Borel measure on a topological space is a measure that is defined on all open sets (and thus on all Borel sets).

distribution function-
The **distribution function** , also called the cumulative **distribution function** (CDF) or cumulative frequency **function**, describes the probability that a variate takes on a value less than or equal to a number .

5.7 QUESTIONS FOR REVIEW

If $-\infty < x_1 < x_2 < \cdots < x_N < +\infty$ and $0 < \lambda_1,\dots,\lambda_N < +\infty$, then find (and draw) the cumulative distribution function of $\mu = \sum_{k=1}^{N} \lambda_k \delta_{x_k}$.

The Cantor measure.

Consider the Cantor function f (exercise 4.6.10) extended to **R** by $f(x) = 0$ for all $x < 0$ and $f(x) = 1$ for all $x > 1$. Then $f : \mathbf{R} \to [0,1]$ is increasing, continuous and bounded.

f is the cumulative distribution function of μ_f.

Prove that $\mu_f(C) = \mu_f(\mathbf{R}) = 1$.

Each one of the 2^n subintervals of I_n (look at the construction of C) has measure equal to $\frac{1}{2^n}$.

Let μ be a locally finite Borel measure on **R** such that $\mu((-\infty,0]) < +\infty$. Prove that there is a unique $f : \mathbf{R} \to \mathbf{R}$ increasing and continuous from the right so that $\mu = \mu_f$ and $f(-\infty) = 0$. Which is this function?

The support of a regular Borel measure.

Let μ be a regular Borel measure on the topological space X. A point $x \in X$ is called **a support point for** μ if $\mu(U_x) > 0$ for every open neighborhood U_x of x. The set $\mathrm{supp}(\mu) = \{x \in X \mid x \text{ is a support point for } \mu\}$

is called **the support of** μ.

Prove that supp(μ) is a closed set in X.

Prove that $\mu(K) = 0$ for all compact sets $K \subseteq (\text{supp}(\mu))^c$.

Using the regularity of μ, prove that $\mu((\text{supp}(\mu))^c) = 0.$

Prove that $(\text{supp}(\mu))^c$ is the largest open set in X which is μ-null.

If f is the Cantor function (exercise 5.5.2), prove that the support (exercise 5.5.6) of μ_f is the Cantor set C.

Restrictions of regular Borel measures.

Let μ be a σ-finite regular Borel measure on the topological space X and Y be a Borel subset of X. Prove that the restriction μ_Y is a regular Borel measure on X.

Continuous regular Borel measures.

Let μ be a regular Borel measure on the topological space X so that $\mu(\{x\}) = 0$ for all $x \in X$. A measure satisfying this last property is called **continuous.** Prove that for every Borel set A in X with $0 < \mu(A) < +\infty$ and every $t \in (0,\mu(A))$ there is some Borel set B in X so that $B \subseteq A$ and $\mu(B) = t$.

Let X be a separable, complete metric space and μ be a Borel measure on X so that $\mu(X) = 1$. Prove that there is some B, a countable union of compact subsets of X, so that $\mu(B) = 1$.

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5.9 ANSWERS TO CHECK YOUR PROGRESS

1.please check section 5.3 for answers .

2.please Check section 5.4 for answers.

CHAPTER 6 MEASURABLE FUNCTIONS

STRUCTURE

6.1 OBJECTIVES

In this chapter we are going to learn about the different functions and arithmetic operations which we can perform on the, measurable functions.

6.2 INTRODUCTION

Let (X,Σ) *and* (Y,Σ^0) *measurable spaces and* $f : X \to Y$ *. Suppose that* E *is a collection of subsets of* Y *so that* $\Sigma(\mathrm{E}) = \Sigma^0$*. If* $f^{-1}(E) \in \Sigma$ *for all* $E \in \mathrm{E}$*, then* f *is* (Σ,Σ^0)*-measurable.*

6.3 MEASURABILITY

Definition 6.1 *Let* (X,Σ) *and* (Y,Σ^0) *be measurable spaces and* $f : X \to Y$ *. We say that* f *is* (Σ,Σ^0)*-measurable if* $f^{-1}(E) \in \Sigma$ *for all* $E \in \Sigma^0$*.*

Example

A constant function is measurable. In fact, let (X,Σ) and (Y,Σ^0) be measurable spaces and $f(x) = y_0 \in Y$ for all $x \in X$. Take arbitrary $E \in \Sigma^0$. If $y_0 \in E$, then $f^{-1}(E) = X \in \Sigma$. If $y_0 \notin E$, then $f^{-1}(E) = \emptyset \in \Sigma$.

Proposition 6.1 *Let* (X,Σ) *and* (Y,Σ^0) *measurable spaces and* $f : X \to Y$ *. Suppose that* E *is a collection of subsets of* Y *so that* $\Sigma(\mathrm{E}) = \Sigma^0$*. If* $f^{-1}(E) \in \Sigma$ *for all* $E \in \mathrm{E}$*, then* f *is* (Σ,Σ^0)*-measurable.*

Proof: We consider the collection $\mathrm{S} = \{E \subseteq Y \mid f^{-1}(E) \in \Sigma\}$.

Since $f^{-1}(\emptyset) = \emptyset \in \Sigma$, it is clear that $\emptyset \in \mathrm{S}$.

If $E \in \mathrm{S}$, then $f^{-1}(E^c) = (f^{-1}(E))^c \in \Sigma$ and thus $E^c \in \mathrm{S}$.

If $E_1,E_2,\ldots \in \mathrm{S}$, then $f^{-1}(\cup_{j=1}^{+\infty} E_j) = \cup_{j=1}^{+\infty} f^{-1}(E_j) \in \Sigma$, implying that $\cup_{j=1}^{+\infty} E_j \in \mathcal{S}$.

Therefore S is a σ-algebra of subsets of Y . E is, by hypothesis, included in S and, thus, $\Sigma^0 = \Sigma(\mathrm{E}) \subseteq \mathrm{S}$. This concludes the proof.

Proposition 6.2 *Let* X,Y *be topological spaces and* $f : X \to Y$ *be continuous on* X*. Then* f *is* $(\mathrm{B}_X,\mathrm{B}_Y)$*-measurable.*

Proof: Let E be the collection of all open subsets of Y . Then, by continuity, $f^{-1}(E)$ is an open and, hence, Borel subset of X for all $E \in \mathrm{E}$. Since $\Sigma(\mathrm{E}) = \mathrm{B}_Y$, Proposition 6.1 implies that f is $(\mathrm{B}_X,\mathrm{B}_Y)$-measurable.

6.4 RESTRICTION AND GLUING.

If $f : X \to Y$ and $A \subseteq X$ is non-empty, then the function $f\mathrm{e}A : A \to Y$, defined by $(f\mathrm{e}A)(x) = f(x)$ for all $x \in A$, is the usual **restriction of** f **on** A.

Recall that, if Σ is a σ-algebra of subsets of X and $A \in \Sigma$ is non-empty, then, by Lemma 2.1, $\Sigma e A = \{E \subseteq A | E \in \Sigma\}$ is a σ-algebra of subsets of A. We call $\Sigma e A$ **the restriction of** Σ **on** A.

Proposition 6.3 *Let* (X,Σ), (Y,Σ^0) *be measurable spaces and* $f : X \to Y$. *Let the non-empty* $A_1,\dots,A_n \in \Sigma$ *be pairwise disjoint and* $A_1 \cup \cdots \cup A_n = X$.

Then f *is* (Σ,Σ^0)*-measurable if and only if* $f e A_j$ *is* $(\Sigma e A_j,\Sigma^0)$*-measurable for all* $j = 1,\dots,n$.

Proof: Let f be (Σ,Σ^0)-measurable. For all $E \in \Sigma^0$ we have $(f e A_j)^{-1}(E) = f^{-1}(E) \cap A_j \in \Sigma e A_j$ because the set $f^{-1}(E) \cap A_j$ belongs to Σ and is included in A_j. Hence $f e A_j$ is $(\Sigma e A_j,\Sigma^0)$-measurable for all j.

Now, let $f e A_j$ be $(\Sigma e A_j,\Sigma^0)$-measurable for all j. For every $E \in \Sigma^0$ we have that $f^{-1}(E) \cap A_j = (f e A_j)^{-1}(E) \in \Sigma e A_j$ and, hence, $f^{-1}(E) \cap A_j \in \Sigma$ for all j. Therefore $f^{-1}(E) = (f^{-1}(E) \cap A_1) \cup \cdots \cup (f^{-1}(E) \cap A_n) \in \Sigma$, implying that f is (Σ,Σ^0)-measurable.

In a free language: *measurability of a function separately on complementary (measurable) pieces of the space is equivalent to measurability on the whole space*.

There are two operations on measurable functions that are taken care of by Proposition 6.3. One is the restriction of a function $f : X \to Y$ on some non-empty $A \subseteq X$ and the other is the **gluing** of functions $f e A_j : A_j \to Y$ to form a single $f : X \to Y$, whenever the finitely many A_j's are non-empty, pairwise disjoint and cover X. The rules are: *restriction of measurable functions on measurable sets are measurable* and *gluing of measurable functions defined on measurable subsets results to a measurable function*.

6.5 FUNCTIONS WITH ARITHMETICAL VALUES.

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Definition 6.2 *Let* (X,Σ) *be measurable space and* $f : X \to$ **R** *or* **R** *or* **C** *or*

—

C. *We say f is Σ-measurable if it is* (Σ,B$_{\mathbf{R}}$ *or* B$_{\overline{\mathbf{R}}}$ *or* B$_{\mathbf{C}}$ *or* B$_{\overline{\mathbf{C}}}$)-*measurable, respectively.*

In the particular case when (X,Σ) *is* $(\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n})$ *or* $(\mathbf{R}^n, \mathcal{L}_n)$, *then we use the term Borel measurable or, respectively, Lebesgue measurable for f.*

If $f : X \to \mathbf{R}$, then it is also true that $f : X \to \overline{\mathbf{R}}$. Thus, according to the definition we have given, there might be a conflict between the two meanings of Σ-measurability of *f*. But, actually, there is no such conflict. Suppose, for example, that *f* is assumed (Σ,B$_{\overline{\mathbf{R}}}$)-measurable. If $E \in$ B$_{\mathbf{R}}$, then $E \cap \mathbf{R} \in$ B$_{\overline{\mathbf{R}}}$ and, thus, $f^{-1}(E) = f^{-1}(E \cap \mathbf{R}) \in \Sigma$. Hence *f* is (Σ,B$_{\mathbf{R}}$)-measurable. Let, conversely, *f* be (Σ,B$_{\mathbf{R}}$)-measurable. If $E \in$ B$_{\mathbf{R}}$, then $E \in$ B$_{\overline{\mathbf{R}}}$ and, thus, $f^{-1}(E) \in \Sigma$. Hence *f* is (Σ,B$_{\overline{\mathbf{R}}}$)-measurable.

The same question arises when $f : X \to \mathbf{C}$, because it is then also true that $f : X \to \overline{\mathbf{C}}$. Exactly as before, we may prove that *f* is (Σ,B$_{\mathbf{C}}$)-measurable if and only if it is (Σ,B$_{\overline{\mathbf{C}}}$)-measurable and there is no conflict in the definition.

Proposition 6.4 *Let* (X,Σ) *be measurable space and* $f : X \to \mathbf{R}^n$. *Let, for each* $j = 1,...,n$, $f_j : X \to \mathbf{R}$ *denote the j-th component function of f. Namely,* $f(x) = (f_1(x),...,f_n(x))$ *for all* $x \in X$.

Then f is (Σ,B$_{\mathbf{R}^n}$)-*measurable if and only if every* f_j *is Σ-measurable.*

Proof: Let *f* be (Σ,B$_{\mathbf{R}^n}$)-measurable. For all intervals $(a,b]$ we have

$$f_j^{-1}((a,b]) = f^{-1}(\mathbf{R} \times \cdots \times \mathbf{R} \times (a,b] \times \mathbf{R} \times \cdots \times \mathbf{R})$$

which belongs to Σ. Since the collection of all $(a,b]$ generates B$_{\mathbf{R}}$, Proposition
6.1 implies that f_j is Σ-measurable.

Now let every f_j be Σ-measurable. Then

$$f^{-1}((a_1,b_1] \times \cdots \times (a_n,b_n]) = f_1^{-1}((a_1,b_1]) \cap \cdots \cap f_n^{-1}((a_n,b_n])$$

which is an element of Σ. The collection of all open-closed intervals generates B$_{\mathbf{R}^n}$ and Proposition 6.1, again, implies that *f* is (Σ,B$_{\mathbf{R}^n}$)-measurable.

In a free language: *measurability of a vector function is equivalent to measurability of all component functions*.

The next two results give simple criteria for measurability of real or complex valued functions.

Proposition 6.5 *Let* (X,Σ) *be measurable space and* $f : X \to \mathbf{R}$. *Then* f *is* Σ-*measurable if and only if* $f^{-1}((a,+\infty)) \in \Sigma$ *for all* $a \in \mathbf{R}$.

Proof: Since $(a,+\infty) \in B_{\mathbf{R}}$, one direction is trivial.

If $f^{-1}((a,+\infty)) \in \Sigma$ for all $a \in \mathbf{R}$, then $f^{-1}((a,b]) = f^{-1}((a,+\infty)) \setminus f^{-1}((b,+\infty)) \in \Sigma$ for all $(a,b]$. Now the collection of all intervals $(a,b]$ generates $B_{\mathbf{R}}$ and Proposition 6.1 implies that f is Σ-measurable.

Of course, in the statement of Proposition 6.5 one may replace the intervals $(a,+\infty)$ by the intervals $[a,+\infty)$ or $(-\infty,b)$ or $(-\infty,b]$.

If $f : X \to \mathbf{C}$, then the functions $<(f),=(f) : X \to \mathbf{R}$ are defined by $<(f)(x) = <(f(x))$ and $=(f)(x) = =(f(x))$ for all $x \in X$ and they are called **the real part** and **the imaginary part of** f, respectively.

Proposition 6.6 *Let* (X,Σ) *be measurable space and* $f : X \to \mathbf{C}$. *Then* f *is* Σ-*measurable if and only if both* $<(f)$ *and* $=(f)$ *are* Σ-*measurable.*

Proof: An immediate application of Proposition 6.4.

The next two results investigate extended-real or extended-complex valued functions.

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Proposition 6.7 *Let* (X,Σ) *be measurable space and* $f : X \to \mathbf{R}$. *The following are equivalent.*

f *is* Σ-*measurable.*

$f^{-1}(\{+\infty\}), f^{-1}(\mathbf{R}) \in \Sigma$ *and, if* $A = f^{-1}(\mathbf{R})$ *is non-empty, the function* $f\text{e}A : A \to \mathbf{R}$ *is* $\Sigma\text{e}A$-*measurable.*

$f^{-1}((a,+\infty]) \in \Sigma$ *for all* $a \in \mathbf{R}$.

Proof: It is trivial that (i) implies (iii), since $(a,+\infty] \in B_{\mathbf{R}}$ for all $a \in \mathbf{R}$.

Assume (ii) and consider $B = f^{-1}(\{+\infty\}) \in \Sigma$ and $C = f^{-1}(\{-\infty\}) = (A \cup B)^c \in \Sigma$. The restrictions $f\text{e}B = +\infty$ and $f\text{e}C = -\infty$ are constants and hence are, respectively, $\Sigma\text{e}B$-measurable and $\Sigma\text{e}C$-measurable. Proposition 6.3 implies that f is Σ-measurable and thus (ii) implies (i).

Now assume (iii). Then $f^{-1}(\{+\infty\}) = \cap_{n=1}^{+\infty} f^{-1}((n,+\infty]) \in \Sigma$ and then $f^{-1}((a,+\infty)) = f^{-1}((a,+\infty]) \setminus f^{-1}(\{+\infty\}) \in \Sigma$ for all $a \in \mathbf{R}$. Moreover, $f^{-1}(\mathbf{R}) = \cup_{n=1}^{+\infty} f^{-1}((-n,+\infty)) \in \Sigma$. For all $a \in \mathbf{R}$ we get $(f\text{e}A)^{-1}((a,+\infty)) =$

$f^{-1}((a,+\infty)) \in \Sigma e A$, because the last set belongs to Σ and is included in A. Proposition 6.5 implies that feA is ΣeA-measurable and (ii) is now proved.

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Proposition 6.8 *Let* (X,Σ) *be measurable space and* $f : X \to \mathbf{C}$*. The following are equivalent.*

f is Σ-measurable.

$f^{-1}(\mathbf{C}) \in \Sigma$ and, if $A = f^{-1}(\mathbf{C})$ is non-empty, the $feA : A \to \mathbf{C}$ is ΣeA-measurable.

Proof: Assume (ii) and consider $B = f^{-1}(\{\infty\}) = (f^{-1}(\mathbf{C}))^c \in \Sigma$. The restriction feB is constant ∞ and hence ΣeB-measurable. Proposition 6.3 implies that f is Σ-measurable. Thus (ii) implies (i).

Now assume (i). Then $A = f^{-1}(\mathbf{C}) \in \Sigma$ since $\mathbf{C} \in$ B$_{\mathbf{C}}$. Proposition 6.3 implies that feA is ΣeA-measurable and (i) implies (ii).

6.6 COMPOSITION.

Proposition 6.9 *Let* (X,Σ)*,* (Y,Σ^0)*,* (Z,Σ^{00}) *be measurable spaces and let* $f : X \to Y$ *,* $g : Y \to Z$*. If* f *is* (Σ,Σ^0)*-measurable and* g *is* (Σ^0,Σ^{00})*-measurable, then* $g \circ f : X \to Z$ *is* (Σ,Σ^{00})*-measurable.*

Proof: For all $E \in \Sigma^{00}$ we have ($(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E)) \in \Sigma$, because $g^{-1}(E) \in \Sigma^0$.

Hence: *composition of measurable functions is measurable*.

6.7 SUMS AND PRODUCTS.

The next result is: *sums and products of real or complex valued measurable functions are measurable functions*.

Proposition 6.10 *Let* (X,Σ) *be a measurable space and* $f,g : X \to \mathbf{R}$ *or* $\mathbf{C}$ *be* Σ*-measurable. Then* $f+g, fg$ *are* Σ*-measurable.*

Proof: (a) We consider $H : X \to \mathbf{R}^2$ by the formula $H(x) = (f(x),g(x))$ for all $x \in X$. Proposition 6.4 implies that H is $(\Sigma$,B$_{\mathbf{R}}2)$-measurable. Now consider $\varphi,\psi : \mathbf{R}^2 \to \mathbf{R}$ by the formulas $\varphi(y,z) = y+z$ and $\psi(y,z) = yz$.

These functions are continuous and Proposition 6.2 implies that they are $(B_{\mathbf{R}}2,B_{\mathbf{R}})$-measurable.

Therefore the compositions $\varphi \circ H, \psi \circ H : X \to \mathbf{R}$ are Σ-measurable. But $(\varphi \circ H)(x) = f(x)+g(x) = (f+g)(x)$ and $(\psi \circ H)(x) = f(x)g(x) = (fg)(x)$ for all $x \in X$ and we conclude that $f+g = \varphi \circ H$ and $fg = \psi \circ H$ are Σ-measurable. (b) In the case $f,g : X \to \mathbf{C}$ we consider $<(f),=(f),<(g),=(g) : X \to \mathbf{R}$, which, by Proposition 6.6, are all Σ-measurable. Then, part (a) implies that $<(f+g) = <(f) + <(g), =(f+g) = =(f) + =(g), <(fg) = <(f)<(g) - =(f)=(g), =(fg) = <(f)=(g) + =(f)<(g)$ are all Σ-measurable. Proposition 6.6 again, gives that $f+g, fg$ are Σ-measurable.

If we want to extend the previous results to functions with infinite values, we must be more careful.

The sums $(+\infty)+(-\infty), (-\infty)+(+\infty)$ are not defined in $\overline{\mathbf{R}}$ and neither is

$\infty + \infty$ defined in $\mathbf{C}$. Hence, when we add $f,g : X \to \mathbf{R}$ or $\mathbf{C}$, we must agree on how to treat the summation on, respectively, the set $B = \{x \in X \mid f(x) = +\infty, g(x) = -\infty \text{ or } f(x) = -\infty, g(x) = +\infty\}$ or the set $B = \{x \in X \mid f(x) = \infty, g(x) = \infty\}$. There are two standard ways to do this. One is to ignore the bad set and consider $f+g$ defined on $B^c \subseteq X$, on which it is naturally defined. The other way is to choose some appropriate h defined on B and define $f+g = h$ on B. The usual choice for h is some constant, e.g $h = 0$.

Proposition 6.11 *Let (X,Σ) be a measurable space and $f,g : X \to \mathbf{R}$ be Σmeasurable. Then the set*

$$B = \{x \in X \mid f(x) = +\infty, g(x) = -\infty \text{ or } f(x) = -\infty, g(x) = +\infty\}$$

belongs to Σ.

The function $f+g : B^c \to \mathbf{R}$ is $\Sigma e B^c$-measurable.

If $h : B \to \mathbf{R}$ is $\Sigma e B$-measurable and we define

$$(f+g)(x) = \begin{cases} f(x)+g(x), & \text{if } x \in B^c, \\ h(x), & \text{if } x \in B, \end{cases}$$

then $f + g : X \to \mathbf{R}$ *is* Σ*-measurable.*

Similar results hold if $f,g : X \to \mathbf{C}$ *and* $B = \{x \in X \,|f(x) = \infty, g(x) = \infty\}$.

Proof: We have

$$B = \left(f^{-1}(\{+\infty\}) \cap g^{-1}(\{-\infty\})\right) \cup \left(f^{-1}(\{-\infty\}) \cap g^{-1}(\{+\infty\})\right) \in \Sigma.$$

Consider the sets $A = \{x \in X \,|f(x), g(x) \in \mathbf{R}\}$, $C_1 = \{x \in X \,|f(x) = +\infty, g(x)$ $6= -\infty$ or $f(x)$ $6= -\infty, g(x) = +\infty\}$ and $C_2 = \{x \in X \,|f(x) = -\infty, g(x)$ $6= +\infty$ or $f(x)$ $6= +\infty, g(x) = -\infty\}$. It is clear that $A, C_1, C_2 \in \Sigma$, that $B^c = A \cup C_1 \cup C_2$ and that the three sets are pairwise disjoint.

The restriction of $f + g$ on A is the sum of the real valued $f{\rm e}A, g{\rm e}A$. By Proposition 6.3, both $f{\rm e}A, g{\rm e}A$ are $\Sigma{\rm e}A$-measurable and, by Proposition 6.10, $(f+g){\rm e}A = f{\rm e}A + g{\rm e}A$ is $\Sigma{\rm e}A$-measurable. The restriction $(f+g){\rm e}C_1$ is constant $+\infty$, and is thus $\Sigma{\rm e}C_1$-measurable. Also the restriction $(f + g){\rm e}C_2$ $= -\infty$ is $\Sigma{\rm e}C_2$-measurable. Proposition 6.3 implies that $f + g : B^c \to \mathbf{R}$ is $\Sigma{\rm e}B^c$measurable.

This is immediate after the result of (i) and Proposition 6.3.

The case $f,g : X \to \mathbf{C}$ is similar, if not simpler.

For multiplication we make the following

Convention: $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$ *in* $\mathbf{R}$ *and* $\infty \cdot 0 = 0 \cdot \infty = 0$ *in* $\mathbf{C}$.

Thus, multiplication is always defined and we may state that: *the product of measurable functions is measurable*.

Proposition 6.12 *Let* (X, Σ) *be a measurable space and* $f,g : X \to \mathbf{R}$ *or* $\mathbf{C}$ *be* Σ*-measurable. Then the function fg is* Σ*-measurable.*

Proof: Let $f,g : X \to \mathbf{R}$.

Consider the sets $A = \{x \in X \,|f(x), g(x) \in \mathbf{R}\}$, $C_1 = \{x \in X \,|f(x) = +\infty, g(x) > 0$ or $f(x) = -\infty, g(x) < 0$ or $f(x) > 0, g(x) = +\infty$ or $f(x) < 0, g(x) = -\infty\}$, $C_2 = \{x \in X \,|f(x) = -\infty, g(x) > 0$ or $f(x) = +\infty, g(x) < 0$ or $f(x) > 0, g(x) = -\infty$ or $f(x) < 0, g(x) = +\infty\}$ and $D = \{x \in X \,|f(x) = \pm\infty, g(x)$

$= 0$ or $f(x) = 0, g(x) = \pm\infty\}$. These four sets are pairwise disjoint, their union is X and they all belong to Σ.

The restriction of fg on A is equal to the product of the real valued feA,geA, which, by Propositions 6.3 and 6.10, is ΣeA-measurable. The restriction (fg)eC_1 is constant $+\infty$ and, hence, ΣeC_1-measurable. Similarly, (fg)e$C_2 = -\infty$ is ΣeC_2-measurable. Finally, (fg)e$D = 0$ is ΣeD-measurable.

Proposition 6.3 implies now that fg is Σ-measurable.

If $f,g : X \to \mathbf{C}$, the proof is similar and slightly simpler.

6.8 ABSOLUTE VALUE AND SIGNUM.

The action of the absolute value on infinities is: $|+\infty| = |-\infty| = +\infty$ and $|\infty| = +\infty$.

Proposition 6.13 *Let* (X,Σ) *be a measurable space and* $f : X \to \mathbf{R}$ *or* $\mathbf{C}$ *be* Σ*-measurable. Then the function* $|f| : X \to [0,+\infty]$ *is* Σ*-measurable.*

Proof: Let $f : X \to \mathbf{R}$. The function $|\cdot| : \mathbf{R} \to [0,+\infty]$ is continuous and, hence, $(B_{\mathbf{R}},B_{\mathbf{R}})$-measurable. Therefore, $|f|$, the composition of $|\cdot|$ and f, is Σ-measurable.

The same proof applies in the case $f : X \to \mathbf{C}$.

Definition 6.3 *For every* $z \in \mathbf{C}$ *we define*

$$sign(z) = \begin{cases} \frac{z}{|z|}, & \text{if } z \neq 0,\infty, \\ 0, & \text{if } z = 0, \\ 0, & \text{if } z = \infty. \end{cases}$$

If we denote $\mathbf{C}^* = \mathbf{C} \setminus \{0,\infty\}$, then the restriction sign$\text{e}\mathbf{C}^* : \mathbf{C}^* \to \mathbf{C}$ is continuous. This implies that, for every Borel set E in $\mathbf{C}$, the set $(\text{signe}\mathbf{C}^*)^{-1}(E)$ is a Borel set contained in $\mathbf{C}^*$. The restriction $\text{sign}\overline{\text{e}}\{0\}$ is constant 0 and the restriction $\text{signe}\{\infty\}$ is constant ∞. Therefore, for every Borel set E in $\mathbf{C}$, the sets $(\text{signe}\{0\})^{-1}(E),(\text{signe}\{\infty\})^{-1}(E)$ are Borel sets. Altogether, $\text{sign}^{-1}(E)$ = $(\text{signe}\mathbf{C}^*)^{-1}(E)\cup(\text{signe}\{0\})^{-1}(E)\cup(\text{signe}\{\infty\})^{-1}(E)$ is a Borel set in $\mathbf{C}$. This

means that sign : $\overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is $(B_{\mathbf{C}},B_{\mathbf{C}})$-measurable.

All this applies in the same way to the function sign : $\mathbf{R} \to \mathbf{R}$ with the simple formula

$$(1, \text{ if } 0 < x \leq +\infty, \ \text{sign}(x) = -1, \text{ if } -\infty \leq x < 0, \ 0, \text{ if } x = 0.$$

Hence sign : $\overline{\mathbf{R}} \to \overline{\mathbf{R}}$ is $(B_{\mathbf{R}},B_{\mathbf{R}})$-measurable.

For all $z \in \mathbf{C}$ we may write

$$z = \text{sign}(z) \cdot |z|$$

and this is called **the polar decomposition of** z.

Proposition 6.14 *Let* (X,Σ) *be a measurable space and* $f : X \to \overline{\mathbf{R}}$ *or* $\overline{\mathbf{C}}$ *be* Σ*-measurable. Then the function sign*(f) *is* Σ*-measurable.*

Proof: If $f : X \to \overline{\mathbf{R}}$, then sign$(f)$ is the composition of sign : $\overline{\mathbf{R}} \to \overline{\mathbf{R}}$ and f and the result is clear by Proposition 6.9. The same applies if $f : X \to \mathbf{C}$.

6.9 MAXIMUM AND MINIMUM.

Proposition 6.15 *Let* (X,Σ) *be measurable space and* $f_1,\ldots,f_n : X \to \overline{\mathbf{R}}$ *be* Σ*-measurable. Then the functions* $\max\{f_1,\ldots,f_n\},\min\{f_1,\ldots,f_n\} : X \to \mathbf{R}$ *are* Σ*-measurable.*

Proof: If $h = \max\{f_1,\ldots,f_n\}$, then for all $a \in \mathbf{R}$ we have $h^{-1}((a,+\infty]) = \cup_{j=1}^{n} f_j^{-1}((a,+\infty]) \in \Sigma$. Proposition 6.7 implies that h is Σ-measurable and from $\min\{f_1,\ldots,f_n\} = -\max\{-f_1,\ldots,-f_n\}$ we see that $\min\{f_1,\ldots,f_n\}$ is also Σ-measurable.

The next result is about *comparison of measurable functions*.

—

Proposition 6.16 *Let* (X,Σ) *be a measurable space and* $f,g : X \to \mathbf{R}$ *be* Σ*measurable. Then* $\{x \in X \mid f(x) = g(x)\}, \{x \in X \mid f(x) < g(x)\} \in \Sigma$.

—

If $f,g : X \to \mathbf{C}$ *is* Σ*-measurable, then* $\{x \in X \mid f(x) = g(x)\} \in \Sigma$.

Proof: Consider the set $A = \{x \in X \mid f(x), g(x) \in \mathbf{R}\} \in \Sigma$. Then the functions $f\text{e}A, g\text{e}A$ are $\Sigma\text{e}A$-measurable and thus $f\text{e}A - g\text{e}A$ is $\Sigma\text{e}A$-measurable. Hence the sets $\{x \in A \mid f(x) = g(x)\} = (f\text{e}A - g\text{e}A)^{-1}(\{0\})$ and $\{x \in A \mid f(x) < g(x)\} = (f\text{e}A - g\text{e}A)^{-1}((-\infty,0))$ belong to $\Sigma\text{e}A$. This, of course, means that these sets belong to Σ (and that they are subsets of A).

We can obviously write $\{x \in X \mid f(x) = g(x)\} = \{x \in A \mid f(x) = g(x)\} \cup$
$\big(f^{-1}(\{-\infty\}) \cap g^{-1}(\{-\infty\})\big) \cup \big(f^{-1}(\{+\infty\}) \cap g^{-1}(\{+\infty\})\big) \in$
Σ. In

a $\{x \in X \mid f(x) < g(x)\} = \{x \in A \mid f(x) < g(x)\} \cup \big(f^{-1}(\{-\infty\}) \cap$

simil

ar

manner, $g^{-1}((-\infty,+\infty]) \cup f^{-1}([-\infty,+\infty)) \cap g^{-1}(\{+\infty\}) \in \Sigma$.

—

The case of $f,g : X \to \mathbf{C}$ and of $\{x \in X \mid f(x) = g(x)\}$ is even simpler.

6.10 TRUNCATION.

There are many possible truncations of a function.

— —

Definition 6.4 *Let* $f : X \to \mathbf{R}$ *and consider* $\alpha,\beta \in \mathbf{R}$ *with* $\alpha \le \beta$. *We define*

$$f_{(\alpha)}^{(\beta)}(x) = \begin{cases} f(x), & \text{if } \alpha \le f(x) \le \beta, \\ \alpha, & \text{if } f(x) < \alpha, \\ \beta, & \text{if } \beta < f(x). \end{cases}$$

We write $f^{(\beta)}$ *instead of* $f_{(-\infty)}^{(\beta)}$ *and* $f_{(\alpha)}$ *instead of* $f_{(\alpha)}^{(+\infty)}$.

The functions $f_{(\alpha)}^{(\beta)}, f^{(\beta)}, f_{(\alpha)}$ *are called truncations of* f.

—

Proposition 6.17 *Let* (X,Σ) *be a measurable space and* $f : X \to \mathbf{R}$ *be a* Σ*-measurable function. Then all truncations* $f_{(\alpha)}^{(\beta)}$ *are* Σ*-measurable.*

Proof: The proof is obvious after the formula $f_{(\alpha)}^{(\beta)} = \min\{\max\{f,\alpha\},\beta\}$.

An important role is played by the following special truncations.

—

Definition 6.5 *Let* $f : X \to \mathbf{R}$. *The* $f^+ : X \to [0,+\infty]$ *and* $f^- : X \to [0,+\infty]$ *defined by the formulas*

$$f^+(x) = \begin{cases} f(x), & \text{if } 0 \le f(x) \\ 0, & \text{if } f(x) < 0, \end{cases} \qquad f^-(x) = \begin{cases} 0, & \text{if } 0 \le f(x) \\ -f(x), & \text{if } f(x) < 0, \end{cases}$$

are called, respectively, the positive part and the negative part of f.

It is clear that $f^+ = f_{(0)}$ and $f^- = -f^{(0)}$. Hence if Σ is a σ-algebra of subsets of X and f is Σ-measurable, then both f^+ and f^- are Σ-measurable. It is also trivial to see that at every $x \in X$ either $f^+(x) = 0$ or $f^-(x) = 0$ and that

$$f^+ + f^- = |f|, \qquad f^+ - f^- = f.$$

There is another type of truncations used mainly for extended-complex valued functions.

— —

Definition 6.6 *Let* $f : X \to \mathbf{R}$ *or* $\mathbf{C}$ *and consider* $r \in [0,+\infty]$. *We define*

$$^{(r)}f(x) = \begin{cases} f(x), & \text{if } |f(x)| \le r, \\ r \cdot sign(f(x)), & \text{if } r < |f(x)|. \end{cases}$$

The functions $^{(r)}f$ *are called truncations of f.*

Observe that, if $f : X \to \mathbf{R}$, then $^{(r)}f = f^{(r)}_{(-r)}$.

— —

Proposition 6.18 *Let* (X,Σ) *be a measurable space and* $f : X \to \mathbf{R}$ *or* $\mathbf{C}$ *a* Σ*-measurable function. Then all truncations* $^{(r)}f$ *are* Σ*-measurable.*

— —

Proof: Observe that the function $\varphi_r : \mathbf{R} \to \mathbf{R}$ with formula

$$\phi_r(x) = \begin{cases} x, & \text{if } |x| \le r \\ r \cdot \text{sign}(x), & \text{if } r < |x|, \end{cases}$$

—

is continuous on $\mathbf{R}$ and hence $(B_{\mathbf{R}},B_{\mathbf{R}})$-measurable. Now $^{(r)}f = \varphi_r \circ f$ is

Σ-measurable.

The proof in the case $f : X \to \mathbf{C}$ is similar.

6.11 LIMITS.

The next group of results is about various *limiting operations on measurable functions*. The rule is, roughly: *the supremum, the infimum and the limit of a sequence of measurable functions are measurable functions*.

Proposition 6.19 *Let* (X,Σ) *be a measurable space and* (f_j) *a sequence of* Σ-

measurable functions $f_j : X \to \mathbf{R}$. *Then all the functions* $\sup_{j\in\mathbf{N}} f_j$, $\inf_{j\in\mathbf{N}} f_j$, $\limsup_{j\to+\infty} f_j$ *and* $\liminf_{j\to+\infty} f_j$ *are* Σ*-measurable.*

Proof: Let $h = \sup_{j\in\mathbf{N}} f_j : X \to \mathbf{R}$. For every $a \in \mathbf{R}$ we have $h^{-1}((a,+\infty]) = \cup_{j=1}^{+\infty} f_j^{-1}((a,+\infty]) \in \Sigma$. Proposition 6.7 implies that h is Σ-measurable.

Now $\inf_{j\in\mathbf{N}} f_j = -\sup_{j\in\mathbf{N}}(-f_j)$ is also Σ-measurable.

And, finally, $\limsup_{j\to+\infty} f_j = \inf_{j\in\mathbf{N}}\left(\sup_{k\geq j} f_k\right)$ and $\liminf_{j\to+\infty} f_j = \sup_{j\in\mathbf{N}}\left(\inf_{k\geq j} f_k\right)$ are Σ-measurable.

Proposition 6.20 *Let* (X,Σ) *be a measurable space and* (f_j) *a sequence of*

Σ*-measurable functions* $f_j : X \to \mathbf{R}$. *Then the set*

$$A = \{x \in X \mid \lim_{j\to+\infty} f_j(x) \text{ exists in } \mathbf{R}\}$$

belongs to Σ.

The function $\lim_{j\to+\infty} f_j : A \to \mathbf{R}$ *is* $\Sigma e A$*-measurable.*

If $h : A^c \to \mathbf{R}$ *is* $\Sigma e A^c$*-measurable and we define*

$$(\lim_{j\to+\infty} f_j)(x) = \begin{cases} \lim_{j\to+\infty} f_j(x), & \text{if } x \in A, \\ h(x), & \text{if } x \in A^c \end{cases}$$

then $\lim_{j\to+\infty} f_j : X \to \mathbf{R}$ *is* Σ*-measurable.*

Similar results hold if $f_j : X \to \mathbf{C}$ *for all* j *and we consider the set* $A = \{x \in X \mid \lim_{j\to+\infty} f_j(x)$ *exists in* $\mathbf{C}\}$.

Proof: (a) Suppose that $f_j : X \to \mathbf{R}$ for all j.

Proposition 6.19 implies that $\limsup_{j\to+\infty} f_j$ and $\liminf_{j\to+\infty} f_j$ are both Σ-measurable. Since $\lim_{j\to+\infty} f_j(x)$ exists if and only if $\limsup_{j\to+\infty} f_j(x) = \liminf_{j\to+\infty} f_j(x)$, we have that

$$A = \{x \in X \mid \limsup_{j\to+\infty} f_j(x) = \liminf_{j\to+\infty} f_j(x)\}$$

and Proposition 6.16 implies that $A \in \Sigma$.

It is clear that the function $\lim_{j\to+\infty} f_j : A \to \mathbf{R}$ is just the restriction of $\limsup_{j\to+\infty} f_j$ (or of $\liminf_{j\to+\infty} f_j$) to A and hence it is ΣeA-measurable.

The proof of (ii) is a direct consequence of (i) and Proposition 6.3.

Let now $f_j : X \to \mathbf{C}$ for all j.

Consider the set $B = \{x \in X \mid \lim_{j\to+\infty} f_j(x)$ exists in $\mathbf{C}\}$ and the set $C = \{x \in X \mid \lim_{j\to+\infty} f_j(x) = \infty\}$. Clearly, $B \cup C = A$.

Now, $C = \{x \in X \mid \lim_{j\to+\infty} |f_j|(x) = +\infty\}$. Since $|f_j| : X \to \mathbf{R}$ for all j, part (a) implies that the function $\lim_{j\to+\infty} |f_j|$ is measurable on the set on which it exists. Therefore, $C \in \Sigma$.

B is the intersection of $B_1 = \{x \in X \mid \lim_{j\to+\infty} <(f_j)(x)$ exists in $\mathbf{R}\}$ and $B_2 = \{x \in X \mid \lim_{j\to+\infty} =(f_j)(x)$ exists in $\mathbf{R}\}$. By part (a) applied to the sequences $(<(f_j)),(=(f_j))$ of real valued functions, we see that the two functions $\lim_{j\to+\infty} <(f_j), \lim_{j\to+\infty} =(f_j)$ are both measurable on the set on which each of them exists. Hence, both B_1, B_2 (the inverse images of $\mathbf{R}$ under these functions) belong to Σ and thus $B = B_1 \cap B_2 \in \Sigma$. Therefore $A = B \cup C \in \Sigma$.

We have just seen that the functions $\lim_{j\to+\infty} <(f_j), \lim_{j\to+\infty} =(f_j)$ are measurable on the set where each of them exists and hence their restrictions to B are both ΣeB-measurable. These functions are, respectively, the real and the imaginary part of the restriction to B of $\lim_{j\to+\infty} f_j$ and Proposition 6.6 says that $\lim_{j\to+\infty} f_j$ is ΣeB-measurable.

Finally, the restriction to C of this limit is constant ∞ and thus it is ΣeC-measurable. By Proposition 6.3, $\lim_{j\to+\infty} f_j$ is ΣeA-measurable.

This is the proof of (i) in the case of complex valued functions and the proof of (ii) is immediate after Proposition 6.3.

—

Finally, let $f_j : X \to \mathbf{C}$ for all j.

For each j we consider the function

$$g_j(x) = \begin{cases} f_j(x), & \text{if } f_j(x) \neq \infty \\ j, & \text{if } f_j(x) = \infty. \end{cases}$$

If we set $A_j = f_j^{-1}(\mathbf{C}) \in \Sigma$, then g_je$A_j = f_j$eA_j is ΣeA_j-measurable. Also g_je$A^c{}_j$ is constant j and hence Σe$A^c{}_j$-measurable. Therefore $g_j : X \to \mathbf{C}$ is Σ-measurable.

It is easy to show that the two limits $\lim_{j\to+\infty} g_j(x)$ and $\lim_{j\to+\infty} f_j(x)$ either both exist or both do not exist and, if they do exist, they are equal. In

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fact, let $\lim_{j\to+\infty} f_j(x) = p \in \mathbf{C}$. If $p \in \mathbf{C}$, then for large enough j we shall have that $f_j(x)$ 6= ∞, implying $g_j(x) = f_j(x)$ and thus $\lim_{j\to+\infty} g_j(x) = p$. If $p = \infty$, then $|f_j(x)| \to +\infty$. Therefore $|g_j(x)| \geq \min\{|f_j(x)|, j\} \to +\infty$ and hence $\lim_{j\to+\infty} g_j(x) = \infty = p$ in this case also. The converse is similarly proved. If $\lim_{j\to+\infty} g_j(x) = p \in \mathbf{C}$, then, for large enough j, $g_j(x)$ 6= j and thus $f_j(x) = g_j(x)$ implying $\lim_{j\to+\infty} f_j(x) = \lim_{j\to+\infty} g_j(x) = p$. If $\lim_{j\to+\infty} g_j(x) = \infty$, then $\lim_{j\to+\infty} |g_j(x)| = +\infty$. Since $|f_j(x)| \geq |g_j(x)|$ we get $\lim_{j\to+\infty} |f_j(x)| = +\infty$ and thus $\lim_{j\to+\infty} f_j(x) = \infty$.

—

Therefore $A = \{x \in X \mid \lim_{j\to+\infty} g_j(x) \text{ exists in } \mathbf{C}\}$ and, applying the result of (b) to the functions $g_j : X \to \mathbf{C}$, we get that $A \in \Sigma$. For the same reason, the function $\lim_{j\to+\infty} f_j$, which on A is equal to $\lim_{j\to+\infty} g_j$, is ΣeA-measurable.

6.12 SIMPLE FUNCTIONS.

Definition 6.7 *Let $E \subseteq X$. The function $\chi_E : X \to \mathbf{R}$ defined by*

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E, \end{cases}$$

is called the characteristic function of E.

Observe that, not only E determines its χ_E, but also χ_E determines the set E by $E = \{x \in X \mid \chi_E(x) = 1\} = \chi_E^{-1}(\{1\})$.

The following are trivial:

$\lambda\chi_E + \kappa\chi_F = \lambda\chi_{E\setminus F} + (\lambda+\kappa)\chi_{E\cap F} + \kappa\chi_{F\setminus E}$ for all $E,F \subseteq X$ and all $\lambda,\kappa \in \mathbf{C}$. $\chi_E\chi_F = \chi_{E\cap F}$ $\quad \chi_{E^c} = 1-\chi_E$

Proposition 6.21 *Let (X,Σ) be a measurable space and $E \subseteq X$. Then χ_E is Σ-measurable if and only if $E \in \Sigma$.*

Proof: If χ_E is Σ-measurable, then $E = \chi_E^{-1}(\{1\}) \in \Sigma$.

Conversely, let $E \in \Sigma$. Then for an arbitrary $F \in \mathcal{B}_{\mathbf{R}}$ or $\mathcal{B}_{\mathbf{C}}$ we have $\chi_E^{-1}(F) = \emptyset$ if $0,1 \notin F$, $\chi_E^{-1}(F) = E$ if $1 \in F$ and $0 \notin F$, $\chi_E^{-1}(F) = E^c$ if $1 \notin F$ and $0 \in F$ and $\chi_E^{-1}(F) = X$ if $0,1 \in F$. In any case, $\chi_E^{-1}(F) \in \Sigma$ and χ_E is Σ-measurable.

Definition 6.8 *A function defined on a non-empty set X is called a simple function on X if its range is a finite subset of* **C**.

The following proposition completely describes the structure of simple functions.

Proposition 6.22 *(i) A function $\varphi : X \to \mathbf{C}$ is a simple function on X if and only if it is a linear combination with complex coefficients of characteristic functions of subsets of X.*

(ii) For every simple function φ on X there are $m \in \mathbf{N}$, different $\kappa_1,\ldots,\kappa_m \in \mathbf{C}$ and non-empty pairwise disjoint $E_1,\ldots,E_m \subseteq X$ with $\cup_{j=1}^m E_j = X$ so that

$$\varphi = \kappa_1\chi_{E_1} + \cdots + \kappa_m\chi_{E_m}.$$

This representation of φ is unique (apart from rearrangement). (iii) If Σ is a σ-algebra of subsets of X, then φ is Σ-measurable if and only if all E_k's in the representation of φ described in (ii) belong to Σ.

Proof: Let $\phi = \sum_{j=1}^n \lambda_j\chi_{F_j}$, where $\lambda_j \in \mathbf{C}$ and $F_j \subseteq X$ for all $j = 1,\ldots,n$. Taking an arbitrary $x \in X$, either x belongs to no F_j, in which case $\varphi(x) = 0$, or, by considering all the sets $F_{j_1},\ldots,F_{j_k}$ which contain x, we have that

$\varphi(x) = \lambda_{j1} + \cdots + \lambda_{jk}$. Therefore the range of φ contains at most all the possible sums $\lambda_{j1} + \cdots + \lambda_{jk}$ together with 0 and hence it is finite. Thus φ is simple on X.

Conversely, suppose φ is simple on X and let its range consist of the different $\kappa_1,\ldots,\kappa_m \in \mathbf{C}$. We consider $E_j = \{x \in X \mid \varphi(x) = \kappa_j\} = \varphi^{-1}(\{\kappa_j\})$. Then every $x \in X$ belongs to exactly one of these sets, so that they are pairwise disjoint and $X = E_1 \cup \cdots \cup E_m$. Now it is clear that $\phi = \sum_{j=1}^{m} \kappa_j \chi_{E_j}$, because both sides take the same value at every x.

If $\phi = \sum_{i=1}^{m'} \kappa'_i \chi_{E'_i}$ is another representation of φ with different $\kappa^0{}_i$'s and nonempty pairwise disjoint E_i^0's covering X, then the range of φ is exactly the set

$\{\kappa'_1, \ldots, \kappa'_{m'}\}$. Hence $m^0 = m$ and, after rearrangement, $\kappa'_1 = \kappa_1, \ldots, \kappa'_m = \kappa_m$. Therefore $E_j^0 = \varphi^{-1}(\{\kappa^0{}_j\}) = \varphi^{-1}(\{\kappa_j\}) = E_j$ for all $j = 1,\ldots,m$. We conclude that the representation is unique.

Now if all E_j's belong to the σ-algebra Σ, then, by Proposition 6.21, all χ_{Ej}'s are Σ-measurable and hence φ is also Σ-measurable. Conversely, if φ is Σ-measurable, then all $E_j = \varphi^{-1}(\{\kappa_j\})$ belong to Σ.

Definition 6.9 *The unique representation of the simple function φ, which is described in part (ii) of Proposition 6.22, is called the standard representation of φ.*

If one of the coefficients in the standard representation of a simple function is equal to 0, then we usually omit the corresponding term from the sum (but then the union of the pairwise disjoint sets which appear in the representation is not, necessarily, equal to the whole space).

Proposition 6.23 *Any linear combination with complex coefficients of simple functions is a simple function and any product of simple functions is a simple function. Also, the maximum and the minimum of real valued simple functions are simple functions.*

Proof: Let φ,ψ be simple functions on X and $p,q \in \mathbf{C}$. Assume that $\lambda_1,\ldots,\lambda_n$ are the values of φ and $\kappa_1,\ldots,\kappa_m$ are the values of ψ. It is obvious that the possible values of $p\varphi + q\psi$ are among the nm numbers $p\lambda_i + q\kappa_j$ and that the possible values of $\varphi\psi$ are among the nm numbers $\lambda_i\kappa_j$. Therefore both functions $p\varphi + q\psi, \varphi\psi$ have a finite number of values. If

φ,ψ are real valued, then the possible values of max$\{\varphi,\psi\}$ and min$\{\varphi,\psi\}$ are among the $n+m$ numbers λ_i,κ_j.

Theorem 6.1 *(i) Given* $\overline{f} : X \to [0,+\infty]$, *there exists an increasing sequence* (φ_n) *of non-negative simple functions on X which converges to f pointwise on X. Moreover, it converges to f uniformly on every subset on which f is bounded. (ii) Given* $f : X \to \mathbf{C}$, *there is a sequence* (φ_n) *of simple functions on X which converges to f pointwise on X and so that* $(|\varphi_n|)$ *is increasing. Moreover,* (φ_n) *converges to f uniformly on every subset on which f is bounded.*

If Σ *is a σ-algebra of subsets of X and f is* Σ*-measurable, then the* φ_n *in (i) and (ii) can be taken to be* Σ*-measurable.*

Proof: (i) For every $n,k \in \mathbf{N}$ with $1 \le k \le 2^{2n}$, we define the sets

$E_n^{(k)} = f^{-1}\Big(\Big(\frac{k-1}{2^n}, \frac{k}{2^n}\Big]\Big), \qquad F_n = f^{-1}((2^n, +\infty])$

and the simple function

$$\phi_n = \sum_{k=1}^{2^{2n}} \frac{k-1}{2^n} \chi_{E_n^{(k)}} + 2^n \chi_{F_n}$$

For each n the sets $E_n^{(1)}, \ldots, E_n^{(2^{2^n})}, F_n$ are pairwise disjoint and their union is the set $f^{-1}((0,+\infty])$, while their complementary set is $G = f^{-1}(\{0\})$. Observe that if f is Σ-measurable then all $E_n^{(k)}$ and F_n belong to Σ and hence φ_n is

Σ-measurable.

In G we have $0 = \varphi_n = f$, in each $E_n^{(k)}$ we have $\phi_n = \frac{k-1}{2^n} < f \le \frac{k}{2^n} = \phi_n + \frac{1}{2^n}$ and in F_n we have $\varphi_n = 2^n < f$.

Now, if $f(x) = +\infty$, then $x \in F_n$ for every n and hence $\varphi_n(x) = 2^n \to +\infty = f(x)$. If $0 \le f(x) < +\infty$, then for all large n we have $0 \le f(x) \le 2^n$ and hence $0 \le f(x) - \phi_n(x) \le \frac{1}{2^n}$, which implies that $\varphi_n(x) \to f(x)$. Therefore, $\varphi_n \to f$ pointwise on X.

If $K \subseteq X$ and f is bounded on K, then there is an n_0 so that $f(x) \le 2^{n0}$ for all $x \in K$. Hence for all $n \ge n_0$ we have $0 \le f(x) - \phi_n(x) \le \frac{1}{2^n}$ for all $x \in K$.

This says that $\varphi_n \to f$ uniformly on K.

It remains to prove that (φ_n) is increasing. If $x \in G$, then $\varphi_n(x) = \varphi_{n+1}(x) = f(x) = 0$. Now observe the relations

$$E_{n+1}^{(2k-1)} \cup E_{n+1}^{(2k)} = E_n^{(k)}, \qquad 1 \le k \le 2^{2n},$$

and

$$(\cup_{l=2^{2n+1}+1}^{2^{2(n+1)}} E_{n+1}^{(l)}) \cup F_{n+1} = F_n.$$

The first relation implies that, if $x \in E_n^{(k)}$ then $\phi_n(x) = \frac{k-1}{2^n}$ and $\varphi_{n+1}(x) = \frac{(2k-1)-1}{2^{n+1}}$ or $\frac{2k-1}{2^{n+1}}$. Therefore, if $x \in E_n^{(k)}$, then $\varphi_n(x) \le \varphi_{n+1}(x)$.

The second relation implies that, if $x \in F_n$, then $\varphi_n(x) = 2^n$ and $\varphi_{n+1}(x) = \frac{(2^{2n+1}+1)-1}{2^{n+1}}$ or ...or $\frac{2^{2(n+1)}-1}{2^{n+1}}$ or 2^{n+1}. Hence, if $x \in F_n$, then $\varphi_n(x) \le \varphi_{n+1}(x)$.

(ii) Let $A = f^{-1}(\mathbf{C})$, whence $f = \infty$ on A^c. Consider the restriction $feA : A \to \mathbf{C}$ and the functions

$$(<(feA))^+,(<(feA))^-,(=(feA))^+,(=(feA))^- : A \to [0,+\infty).$$

If f is Σ-measurable, then $A \in \Sigma$ and these four functions are ΣeA-measurable.

By the result of part (i) there are increasing sequences (p_n), (q_n), (r_n) and (s_n) of non-negative (real valued) simple functions on A so that each converges to, respectively, $(<(feA))^+$, $(<(feA))^-$, $(=(feA))^+$ and $(=(feA))^-$ pointwise on A and uniformly on every subset of A on which feA is bounded (because on such a subset all four functions are also bounded). Now it is obvious that, if we set $\varphi_n = (p_n - q_n) + i(r_n - s_n)$, then φ_n is a simple function on A which is ΣeA-measurable if f is Σ-measurable. It is clear that $\varphi_n \to feA$ pointwise on A and uniformly on every subset of A on which feA is bounded.

Also $|\varphi_n| = \sqrt{(p_n - q_n)^2 + (r_n - s_n)^2} = \sqrt{p_n^2 + q_n^2 + r_n^2 + s_n^2}$ and thus the sequence $(|\varphi_n|)$ is increasing on A.

If we define φ_n as the constant n on A^c, then the proof is complete.

6.13 THE ROLE OF NULL SETS.

Definition 6.10 *Let (X,Σ,μ) be a measure space. We say that a property $P(x)$ holds (μ-)almost everywhere on X or for (μ-)almost every $x \in X$, if the set $\{x \in X \,|P(x)$ is not true$\}$ is included in a (μ-)null set.*

We also use the short expressions: $P(x)$ *holds* $(\mu$-*)a.e. on* X and $P(x)$ *holds for* $(\mu$-*)a.e.* $x \in X$.

It is obvious that if $P(x)$ holds for a.e. $x \in X$ and μ is *complete* then the set $\{x \in X \mid P(x)$ is not true$\}$ is contained in Σ and hence its complement $\{x \in X \mid P(x)$ is true$\}$ is also in Σ.

Proposition 6.24 *Let* (X,Σ,μ) *be a measure space and* $(X,\overline{\Sigma},\overline{\mu})$ *be its completion. Let* (Y,Σ^0) *be a measurable space and* $f : X \to Y$ *be* (Σ,Σ^0)-*measurable.*

If $g : X \to Y$ *is equal to* f *a.e on* X*, then* g *is* $(\overline{\Sigma},\Sigma^0)$-*measurable.*

Proof: There exists $N \in \Sigma$ so that $\{x \in X \mid f(x) \neq g(x)\} \subseteq N$ and $\mu(N) = 0$.

Take an arbitrary $E \in \Sigma^0$ and write $g^{-1}(E) = \{x \in X \mid g(x) \in E\} = \{x \in N^c \mid g(x) \in E\} \cup \{x \in N \mid g(x) \in E\} = \{x \in N^c \mid f(x) \in E\} \cup \{x \in N \mid g(x) \in E\}$.

The first set is $= N^c \cap f^{-1}(E)$ and belongs to Σ and the second set is $\subseteq N$.

By the definiton of the completion we get that $g^{-1}(E) \in \overline{\Sigma}$ and hence g is $(\overline{\Sigma},\Sigma^0)$-measurable.

In the particular case of a *complete* measure space (X,Σ,μ) we have the rule: *if* f *is measurable on* X *and* g *is equal to* f *a.e. on* X*, then* g *is also measurable on* X.

Proposition 6.25 *Let* (X,Σ,μ) *be a measure space and* $(X,\overline{\Sigma},\overline{\mu})$ *be its completion. Let* (f_j) *be a sequence of* Σ-*measurable functions* $f_j : X \to \mathbf{R}$ *or* $\mathbf{C}$. *If* $g : X \to \overline{\mathbf{R}}$ *or* $\mathbf{C}$ *is such that* $g(x) = \lim_{j\to+\infty} f_j(x)$ *for a.e.* $x \in X$*, then* g *is* $\overline{\Sigma}$-*measurable.*

Proof: $\{x \in X \mid \lim_{j\to+\infty} f_j(x)$ does not exist or is $\neq g(x)\} \subseteq N$ for some $N \in \Sigma$ with $\mu(N) = 0$.

N^c belongs to Σ and the restrictions $f_j\restriction N^c$ are all $\Sigma\restriction N^c$-measurable. By Proposition 6.20, the restriction $g\restriction N^c = \lim_{j\to+\infty} f_j\restriction N^c$ is $\Sigma\restriction N^c$-measurable.

This, of course, means that for every $E \in \Sigma^0$ we have $\{x \in N^c \,|\, g(x) \in E\} \in \Sigma$.

Now we write $g^{-1}(E) = \{x \in N^c \,|\, g(x) \in E\} \cup \{x \in N \,|\, g(x) \in E\}$. The first set belongs to Σ and the second is $\subseteq N$. Therefore $g^{-1}(E) \in \overline{\Sigma}$ and g is $\overline{\Sigma}$-measurable.

Again, in the particular case of a *complete* measure space (X,Σ,μ) the rule is: *if* (f_j) *is a sequence of measurable functions on X and its limit is equal to g a.e. on X, then g is also measurable on X.*

Proposition 6.26 *Let* (X,Σ,μ) *be a measure space and* $(X,\overline{\Sigma},\overline{\mu})$ *be its completion. Let* (Y,Σ^0) *be a measurable space and* $f : A \to Y$ *be* $(\Sigma e A,\Sigma^0)$*-measurable, where* $A \in \Sigma$ *with* $\mu(A^c) = 0$. *If we extend f to X in an arbitrary manner, then the extended function is* $(\overline{\Sigma},\Sigma^0)$*-measurable.*

Proof: Let $h : A^c \to Y$ be an arbitrary function and let

$$F(x) = \begin{cases} f(x), & \text{if } x \in A, \\ h(x), & \text{if } x \in A^c \end{cases}.$$

Take an arbitrary $E \in \Sigma^0$ and write $F^{-1}(E) = \{x \in A | f(x) \in E\} \cup \{x \in A^c \,|\, h(x) \in E\} = f^{-1}(E) \cup \{x \in A^c \,|\, h(x) \in E\}$. The first set belongs to $\Sigma e A$ and hence to Σ, while the second set is $\subseteq A^c$. Therefore $F^{-1}(E) \in \overline{\Sigma}$ and F is $(\overline{\Sigma},\Sigma^0)$-measurable.

If (X,Σ,μ) is a *complete* measure space, the rule is: *if f is defined a.e. on X and it is measurable on its domain of definition, then any extension of f on X is measurable*.

Check your progress

Let (X,Σ) be a measurable space and $f : X \to \mathbf{R}$ or $\mathbf{C}$ be measurable. We agree that $0^p = +\infty, (+\infty)^p = 0$ if $p < 0$ and $0^0 = (+\infty)^0 = 1$. Prove that, for all $p \in \mathbf{R}$, the function $|f|^p$ is measurable.

Prove that every monotone $f : \mathbf{R} \to \mathbf{R}$ is Borel measurable.

Translates and dilates of functions.

Let $f : \mathbf{R}^n \to Y$ and take arbitrary $y \in \mathbf{R}^n$ and $\lambda \in (0,+\infty)$. We define $g,h : \mathbf{R}^n \to Y$ by

$$g(x) = f(x-y), \qquad h(x) = f\left(\frac{x}{\lambda}\right)$$

for all $x \in \mathbf{R}^n$. g is called **the translate of** f **by** y and h is called **the dilate of** f **by** λ.

Let (Y,Σ^0) be a measurable space. Prove that, if f is (L_n,Σ^0)-measurable, then the same is true for g and h.

Functions with prescribed level sets.

Let (X,Σ) be a measurable space and assume that the collection $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ of subsets of X, which belong to Σ, has the properties: (i) $E_\lambda \subseteq E_\kappa$ for all λ,κ with $\lambda \le \kappa$,

$\cup_{\lambda\in\mathbf{R}} E_\lambda = X$, $\cap_{\lambda\in\mathbf{R}} E_\lambda = \emptyset$,

$\cap_{\kappa,\kappa>\lambda} E_\kappa = E_\lambda$ for all $\lambda \in \mathbf{R}$.

Consider the function $f : X \to \mathbf{R}$ defined by $f(x) = \inf\{\lambda \in \mathbf{R} | x \in E_\lambda\}$.

Prove that f is measurable and that $E_\lambda = \{x \in X \,|f(x) \le \lambda\}$ for every $\lambda \in \mathbf{R}$.

How will the result change if we drop any of the assumptions in (ii) and (iii)?

Not all functions are Lebesgue measurable and not all Lebesgue measurable functions are Borel measurable.

Prove that a Borel measurable $g : \mathbf{R} \to \mathbf{R}$ is also Lebesgue measurable. (ii) Find a function $f : \mathbf{R} \to \mathbf{R}$ which is not Lebesgue measurable. (iii) Using exercise 4.6.15, find a function $g : \mathbf{R} \to \mathbf{R}$ which is Lebesgue measurable but not Borel measurable.

Give an example of a non-Lebesgue measurable $f : \mathbf{R} \to \mathbf{R}$ so that $|f|$ is Lebesgue measurable.

Starting with an appropriate non-Lebesgue measurable function, give anexample of an uncountable collection $\{f_i\}_{i\in I}$ of Lebesgue measurable functions $f_i : \mathbf{R} \to \mathbf{R}$ so that $\sup_{i\in I} f_i$ is non-Lebesgue measurable.

(i) Prove that, if $G : \mathbf{R} \to \mathbf{R}$ is continuous and $H : \mathbf{R} \to \mathbf{R}$ is Borel measurable, then $H \circ G : \mathbf{R} \to \mathbf{R}$ is Borel measurable.

(ii) Using exercise 4.6.15, construct a continuous $G : \mathbf{R} \to \mathbf{R}$ and a Lebesgue measurable $H : \mathbf{R} \to \mathbf{R}$ so that $H \circ G : \mathbf{R} \to \mathbf{R}$ is not Lebesgue measurable.

— —

Let (X,Σ,μ) be a measure space and $f : X \to \mathbf{R}$ or $\mathbf{C}$ be measurable. Assume that $\mu(\{x \in X \| f(x)| = +\infty\}) = 0$ and that there is $M < +\infty$ so that $\mu(\{x \in X \| f(x)| > M\}) < +\infty$.

6.14 LET US SUM UP

In this unit we discussed the following

Measurability

Restriction and gluing

Functions with arithmetical values.

Composition

Sums and products

Absolute value and signum

Maximum and minimum.

Truncation

Limits

Simple functions

The role of null sets

6.15 KEYWORDS

Measurable space -measurable space or Borel **space** is a basic object in **measure theory**. It consists of a set and a σ-algebra, which defines the subsets that will be measured.

Truncate -**Truncating** is a method of approximating a decimal number by dropping all decimal places past a certain point without rounding. Example: 3.14159265 can be **truncated** to 3.1415.

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6.18. ANSWERS TO CHECK YOUR PROGRESS

1. Check section 6.3-6.13 for answers

UNIT 7 CANTOR TERNARY SET

STRUCTURE

7.1 OBJECTIVE

In this chapter we are going to learn about cantor ternary set and its functions.

We will be learning about the devil's staircase and seeing problems related to it

7.2 INTRODUCTION.

Georg Cantor (1845-1918) introduced the notion of the cantor set, which consists of points along a single line segment with a number of remarkable and deep properties. This paper aims to emphasize a proceeding to obtain the Cantor (ternary) set, C by means of the so-called elimination of the open-middle third at each step using a general basic approach in constructing the set.

Since $\{x\}$ is a degenerate interval, we see that $m_n(\{x\}) = \mathrm{vol}_n(\{x\}) = 0$. In fact, every countable subset of $\mathbf{R}^n$ has Lebesgue measure zero: if $A = \{x_1,x_2,...\}$, then $m_n(A) = \sum_{k=1}^{+\infty} m_n(\{x_k\}) = 0.$

The aim of this section is to provide an uncountable set in **R** whose Lebesgue measure is zero.

We start with the interval

$I_0 = [0,1]$,

then take $$I_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right],$$

next $$I_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$,

and so on, each time dividing each of the intervals we get at the previous stage into three subintervals of equal length and keeping only the two closed subintervals on the sides.

Therefore, we construct a decreasing sequence (I_n) of closed sets so that every I_n consists of 2^n closed intervals all of which have the same length $\frac{1}{3^n}$. We define

$C = \cap_{n=1}^{+\infty} I_n$

and call it **the Cantor set**.

C is a compact subset of $[0,1]$ with $m_1(C) = 0$. To see this observe that for every n, $m_1(C) \leq m_1(I_n) = 2^n \cdot \frac{1}{3^n}$ which tends to 0 as $n \to +\infty$.

We shall prove by contradiction that C is uncountable. Namely, assume that $C = \{x_1, x_2, ...\}$. We shall describe an inductive process of picking one from the subintervals constituting each I_n.

It is obvious that every x_n belongs to I_n, since it belongs to C. At the first step choose the interval $I^{(1)}$ to be the subinterval of I_1 which does *not* contain x_1. Now, $I^{(1)}$ includes two subintervals of I_2 and at the second step choose the interval $I^{(2)}$ to be whichever of these two subintervals of $I^{(1)}$ does *not* contain x_2. (If both do not contain x_2, just take the left one.) And continue inductively: if you have already chosen $I^{(n-1)}$ from the subintervals of I_{n-1}, then this includes two subintervals of I_n. Choose as $I^{(n)}$ whichever of these two subintervals of

$I^{(n-1)}$ does *not* contain x_n. (If both do not contain x_n, just take the left one.) This produces a sequence $(I^{(n)})$ of intervals with the following properties:

(i) $I^{(n)} \subseteq I_n$ for all n,

(ii) $I(n) \subseteq I(n-1)$ for all n,

(iii) vol $_1(I^{(n)}) = \frac{1}{3^n} \to 0$ and

(iv) $x_n \in\!\!/\ I^{(n)}$ for all n.

From (ii) and (iii) we conclude that the intersection of all $I^{(n)}$'s contains a single point:

$\cap_{n=1}^{+\infty} I^{(n)} = \{x_0\}$

for some x_0. From (i) we see that $x_0 \in I_n$ for all n and thus $x_0 \in C$. Therefore, $x_0 = x_n$ for some $n \in \mathbf{N}$. But then $x_0 \in I^{(n)}$ and, by (iv), *the same* point x_n does not belong to $I^{(n)}$.

We get a contradiction and, hence, C is uncountable.

Another way to define cantor set;

Step 0: we begin with the interval [0,1].

Step 1: we divide [0,1] into 3 subintervals and delete the *open* middle subinterval $(\frac{1}{3}, \frac{2}{3})$.

Step 2: we divide each of the 2 resulting intervals above into 3 subintervals and delete the *open* middle subintervals $(\frac{1}{9}, \frac{2}{9})$and $(\frac{7}{9}, \frac{8}{9})$.

We continue this procedure indefinitely. At each step, we delete the *open* middle third subinterval of each interval obtained in the previous step.

Definition of Cantor's Set

Cantor's set is the set C left after this procedure of deleting the open middle third subinterval is performed infinitely many times.

- Is there anything left?

Yes, at least the endpoints of the deleted middle third subintervals.

There are countably many such points.

▪ Are there any *other* points left?

Yes, in some sense, a whole lot more. But in some other sense, just some dust - which in some ways is *scattered*, in some other ways it is *bound together*.

We will describe different ways to"measure" the dust left. This will take us through several mathematical disciplines: set theory, measure theory, topology, geometric measure theory, real analysis.

Some Properties of the Cantor Set

1. The cantor set has no interval
2. The cantor set is non-empty
3. The cantor set is closed and nowhere dense
4. The cantor set is compact
5. The cantor set is perfect and totally disconnected
6. The cantor set is uncountable

7.3 TERNARY REPRESENTATION OF CANTOR'S SET

▪ Every real number can be represented by an infinite sequence of digits:

$$\frac{1}{3} = 0.33333\ldots$$

$$\text{golden ratio} = 1.6180339887498948482045\ldots$$

$$\frac{1}{10} = 0.10000\ldots = 0.09999\ldots$$

This is the decimal (base 10) representation:

Numbers described using powers of 10 and

Digits used: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Some numbers can be represented in two ways, with one representation having only 9's from some point on. ▪ Computers use the binary (base 2) representation: every number is described using powers of 2 and digits 0, 1. golden ratio = $1.10011110001101110111\ldots_{(2)}$

$$\frac{1}{2} = 0.1000\ldots = 0.011111\ldots_{(2)}$$

Ternary Representation of Cantor's Set

We can represent real numbers in any base. We will use the

—

ternary (base 3) representation, because Cantor's set has a special representation in base 3.

$$\frac{1}{3} = 1 \cdot 3^{-1} = 0.10000\ldots_{(3)} = 0.022222\ldots_{(3)}$$
$$\frac{2}{3} = 2 \cdot 3^{-1} = 0.20000\ldots_{(3)}$$
$$\frac{7}{9} = 2 \cdot 3^{-1} + 1 \cdot 3^{-2} = 0.210000\ldots_{(3)} = 0.20222\ldots_{(3)}$$
$$\frac{8}{9} = 2 \cdot 3^{-1} + 2 \cdot 3^{-2} = 0.220000\ldots_{(3)}$$

A number is in Cantor's set if and only if its ternary representation contains only the digits 0 and 2 (in other words, it has no 1's).

$C = \{x \in [0,1]: x = 0.c_1c_2c_3 \ldots c_n \ldots_{(3)}$	where $c_n = 0$ or 2}

6.16 QUESTIONS FOR REVIEW

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Let (X,Σ) be a measurable space and $f : X \to \mathbf{R}$. Prove that f is measurable if $f^{-1}((a,+\infty]) \in \Sigma$ for all *rational* $a \in \mathbf{R}$.

— —

Let $f : X \to \mathbf{R}$. If $g,h : X \to \mathbf{R}$ are such that $g,h \geq 0$ and $f = g - h$ on X, prove that $f^+ \leq g$ and $f^- \leq h$ on X.

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Prove that for every > 0 there is a *bounded* measurable $g : X \to \mathbf{R}$ or $\mathbf{C}$ so that $\mu(\{x \in X \mid g(x) \neq f(x)\}) < \epsilon$. You may try a suitable truncation of f.

We say that $\varphi : X \to \mathbf{C}$ is an **elementary function on** X if it has countably many values. Is there a standard representation for an elementary function?

Prove that for any $f : X \to [0,+\infty)$, there is an increasing sequence (φ_n) of elementary functions on X so that $\varphi_n \to f$ uniformly on X. If Σ is a σ-algebra of subsets of X and f is measurable, prove that the φ_n's can be taken measurable.

We can add, multiply and take limits of equalities holding almost everywhere.

Let (X,Σ,μ) be a measure space.

Let $f,g,h : X \to Y$. If $f = g$ a.e. on X and $g = h$ a.e. on X, then $f = h$ a.e. on X.

Let $f_1,f_2,g_1,g_2 : X \to \mathbf{R}$. If $f_1 = f_2$ a.e. on X and $g_1 = g_2$ a.e. on X, then $f_1 + g_1 = f_2 + g_2$ and $f_1g_1 = f_2g_2$ a.e. on X.

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Let $f_j,g_j : X \to \mathbf{R}$ so that $f_j = g_j$ a.e. on X for all $j \in \mathbf{N}$. Then $\sup_{j\in\mathbf{N}} f_j = \sup_{j\in\mathbf{N}} g_j$ a.e. on X. Similar results hold for inf,limsup and liminf.

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Let $f_j,g_j : X \to \mathbf{R}$ so that $f_j = g_j$ a.e. on X for all $j \in \mathbf{N}$. If $A = \{x \in X \mid \lim_{j\to+\infty} f_j(x)$ exists$\}$ and $B = \{x \in X \mid \lim_{j\to+\infty} g_j(x)$ exists$\}$, then $A4B \subseteq N$ for some $N \in \Sigma$ with $\mu(N) = 0$ and $\lim_{j\to+\infty} f_j = \lim_{j\to+\infty} g_j$ a.e. on $A \cap B$. If, moreover, we extend both $\lim_{j\to+\infty} f_j$ and $\lim_{j\to+\infty} g_j$ by a common function h on $(A\cap B)^c$, then $\lim_{j\to+\infty} f_j = \lim_{j\to+\infty} g_j$ a.e. on X.

Let (X,Σ,μ) be a measure space and (X,Σ,μ) be its completion.

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If $E \in \Sigma$, then there is $A \in \Sigma$ so that $\chi_E = \chi_A$ a.e. on X.

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If $\varphi : X \to \mathbf{C}$ is a Σ-measurable simple function, then there is a Σmeasurable simple function $\psi : X \to \mathbf{C}$ so that $\varphi = \psi$ a.e. on X. (iii) Use Theorem 6.1 to prove that, if $g : X \to \mathbf{R}$ or $\mathbf{C}$ is Σ-measurable, then there is a Σ-measurable $f : X \to \mathbf{R}$ or $\mathbf{C}$ so that $g = f$ a.e. on X.

Let X,Y be topological spaces of which Y is Hausdorff. This means that, if $y_1,y_2 \in Y$ and y_1 6= y_2, then there are *disjoint* open neighborhoods V_{y_1},V_{y_2} of y_1,y_2, respectively. Assume that μ is a Borel measure on X so that $\mu(U) > 0$ for every non-empty open $U \subseteq X$. Prove that, if $f,g : X \to Y$ are continuous and $f = g$ a.e. on X, then $f = g$ on X.

The support of a function.

Let X be a topological space and a continuous $f : X \to \mathbf{C}$. The set

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$\mathrm{supp}(f) = f^{-1}(\mathbf{C} \setminus \{0\})$ is called **the support of** f. Prove that $\mathrm{supp}(f)$ is the smallest closed subset of X outside of which $f = 0$.

Let μ be a regular Borel measure on the topological space X and $f : X \to \mathbf{C}$ be a Borel measurable function. A point $x \in X$ is called **a support point for** f if $\mu(\{y \in U_x | f(y) \neq 0\}) > 0$ for every open neighborhood U_x of x. The set $\text{supp}(f) = \{x \in X | x \text{ is a support point for } f\}$
is called **the support of** f.

Prove that $\text{supp}(f)$ is a closed set in X.

Prove that $\mu(\{x \in K | f(x) \neq 0\}) = 0$ for all compact sets $K \subseteq (\text{supp}(f))^c$.

Using the regularity of μ, prove that $f = 0$ a.e on $(\text{supp}(f))^c$. (iv) Prove that $(\text{supp}(f))^c$ is the largest open set in X on which $f = 0$ a.e.

(c) Assume that the μ appearing in (b) has the additional property that $\mu(U) > 0$ for every open $U \subseteq X$. Use exercise 6.12.15 to prove that for any continuous $f : X \to \mathbf{C}$ the two definitions of $\text{supp}(f)$ (the one in (a) and the one in (b)) coincide.

The Theorem of Lusin.

We shall prove that every Lebesgue measurable function which is finite a.e. on $\mathbf{R}^n$ is equal to a continuous function except on a set of arbitrarily small Lebesgue measure.

For each $a < a+\delta < b-\delta < b$ we consider the function $\tau_{a,b,\delta} : \mathbf{R} \to \mathbf{R}$ which: is 0 outside (a,b), is 1 on $[a+\delta, b-\delta]$ and is linear on $[a, a+\delta]$ and on $[b-\delta, b]$ so that it is continuous on $\mathbf{R}$. Now, let $R = (a_1,b_1)\times\cdots\times(a_n,b_n)$ and, for small enough $\delta > 0$, we consider the function $\tau_{R,\delta} : \mathbf{R}^n \to \mathbf{R}$ by the formula

$$\tau_{R,\delta}(x_1,\dots,x_n) = \tau_{a_1,b_1,\delta}(x_1)\cdots\tau_{a_n,b_n,\delta}(x_n).$$

If $R_\delta = (a_1 + \delta, b_1 - \delta) \times \cdots \times (a_n + \delta, b_n - \delta)$, prove that $\tau_{R,\delta} = 1$ on $\overline{R_\delta}$, $\tau_{R,\delta} = 0$ outside R, $0 \leq \tau_{R,\delta} \leq 1$ on $\mathbf{R}^n$ and $\tau_{R,\delta}$ is continuous on $\mathbf{R}^n$. Therefore, prove that for every $\epsilon > 0$ there is $\delta > 0$ so that

$$m_n(\{x \in \mathbf{R}^n \mid \tau_{R,\delta}(x) \neq \chi_R(x)\}) < \epsilon.$$

Let $E \in \mathcal{L}_n$ with $m_n(E) < +\infty$. Use Theorem 4.6 to prove that for every $\epsilon > 0$ there is a continuous $\tau : \mathbf{R}^n \to \mathbf{R}$ so that $0 \leq \tau \leq 1$ on $\mathbf{R}^n$ and $m_n(\{x \in \mathbf{R}^n \mid \tau(x) \neq \chi_E(x)\}) < \epsilon$.

Let φ be a non-negative Lebesgue measurable simple function on $\mathbf{R}^n$ which is 0 outside some set of finite Lebesgue measure. Prove that for all

> 0 there is a continuous $\tau : \mathbf{R}^n \to \mathbf{R}$ so that $0 \le \tau \le \max_{\mathbf{R}^n} \varphi$ on $\mathbf{R}^n$ and $m_n(\{x \in \mathbf{R}^n \,|\, \tau(x) \neq \phi(x)\}) < \epsilon$.

Let $f : \mathbf{R}^n \to [0,1]$ be a Lebesgue measurable function which is 0 outside some set of finite Lebesgue measure. Use Theorem 6.1 to prove that $f = \sum_{k=1}^{+\infty} \psi_k$ uniformly on $\mathbf{R}^n$, where all ψ_k are Lebesgue measurable simple functions with $0 \le \psi_k \le \frac{1}{2^k}$ on $\mathbf{R}^n$ for all k. Now apply the result of (iii) to each ψ_k and prove that for all > 0 there is a continuous $g : \mathbf{R}^n \to [0,1]$ so that $m_n(\{x \in \mathbf{R}^n \,|\, g(x) \neq f(x)\}) < \epsilon$.

Let $f : \mathbf{R}^n \to [0,+\infty]$ be a Lebesgue measurable function which is 0 outside some set of finite Lebesgue measure and finite a.e. on $\mathbf{R}^n$. By taking an appropriate truncation of f prove that for all > 0 there is a bounded Lebesgue measurable function $h : \mathbf{R}^n \to [0,+\infty]$ which is 0 outside some set of finite Lebesgue measure so that $m_n(\{x \in \mathbf{R}^n \,|\, h(x) \neq f(x)\}) < \epsilon$. Now apply the result of (iv) to find a continuous $g : \mathbf{R}^n \to \mathbf{R}$ so that $m_n(\{x \in \mathbf{R}^n \,|\, g(x) \neq f(x)\}) < \epsilon$.

Find pairwise disjoint open-closed qubes $P^{(k)}$ so that $\mathbf{R}^n = \cup_{k=1}^{+\infty} P^{(k)}$ and let $R^{(k)}$ be the open qube with the same edges as $P^{(k)}$. Consider for each k a small enough $\delta_k > 0$ so that $m_n(\{x \in \mathbf{R}^n \,|\, \tau_{R^{(k)},\delta_k}(x) \neq \chi_{R^{(k)}}(x)\}) < \frac{\epsilon}{2^{k+1}}$.

Let $f : \mathbf{R}^n \to [0,+\infty]$ be Lebesgue measurable and finite a.e. on $\mathbf{R}^n$. If $R^{(k)}$ are the qubes from (vi), then each $f\chi_{R^{(k)}} : \mathbf{R}^n \to [0,+\infty]$ is Lebesgue measurable, finite a.e. on $\mathbf{R}^n$ and 0 outside $R^{(k)}$. Apply (v) to find continuous $g_k : \mathbf{R}^n \to \mathbf{R}$ so that $m_n(\{x \in \mathbf{R}^n \,|\, g_k(x) \neq f(x)\chi_{R^{(k)}}(x)\}) < \frac{\epsilon}{2^{k+1}}$.

Prove that $m_n(\{x \in \mathbf{R}^n \,|\, \tau_{R^{(k)},\delta_k}(x) g_k(x) \neq f(x)\chi_{R^{(k)}}(x)\}) < \frac{\epsilon}{2^k}$.

Define $g = \sum_{k=1}^{+\infty} \tau_{R^{(k)},\delta_k} g_k$ and prove that g is continuous on $\mathbf{R}^n$ and that

$$m_n(\{x \in \mathbf{R}^n \,|\, g(x) \neq f(x)\}) < \epsilon.$$

— —

Extend the result of (vii) to all $f : \mathbf{R}^n \to \mathbf{R}$ or $\mathbf{C}$ which are Lebesgue measurable and finite a.e. on $\mathbf{R}^n$.

Let $f : \mathbf{R}^n \to \mathbf{R}$ be continuous at m_n-a.e. $x \in \mathbf{R}^n$. Prove that f is Lebesgue measurable on $\mathbf{R}^n$.

7.4 CANTOR'S FUNCTION

The function defined earlier, $f : C \rightarrow [0,1]$

$$f(0.c_1c_2c_3\ldots c_n\ldots_{(3)}) := 0.\frac{c_1}{2}\frac{c_2}{2}\frac{c_3}{2}\ldots\frac{c_n}{2}\ldots_{(2)}$$

has the following properties:

- It is onto.
- It is increasing
- It is <u>not</u> one to one. For instance:

$$f\left(\frac{1}{3}\right) = f(0.0222\ldots_{(3)}) = 0.01111\ldots_{(2)} = 0.1_{(2)} = \frac{1}{2}$$

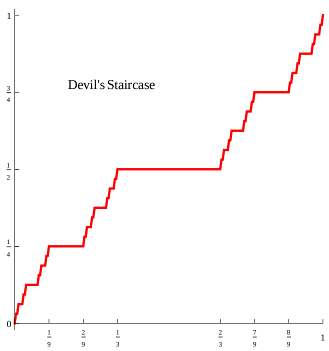
$$f\left(\frac{2}{3}\right) = f(0.2000\ldots_{(3)}) = 0.10000\ldots_{(2)} = 0.1_{(2)} = \frac{1}{2}$$

Two inputs of f have the same outputs if and only if they are the endpoints of an interval removed - like $\left(\frac{1}{3}, \frac{2}{3}\right)$ or $\left(\frac{1}{9}, \frac{2}{9}\right)$ etc.

BACK TO REAL ANALYSIS ; CANTOR'S FUNCTION

Extend f to the whole interval [0,1] by making it constant on these removed intervals.

The function obtained by this extension is called Cantor's function.



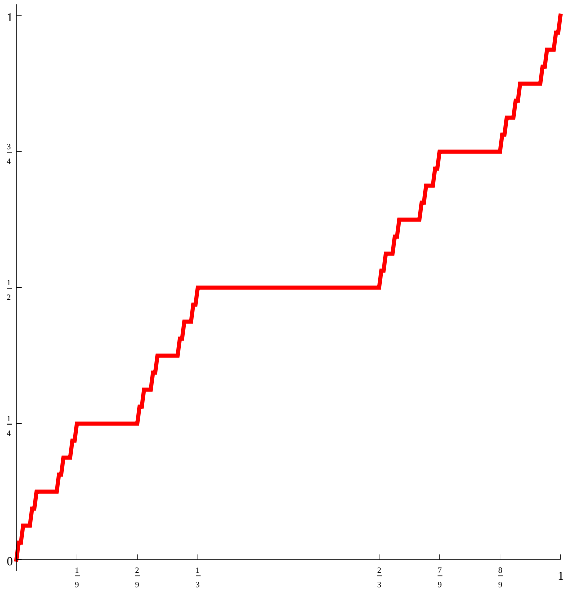
Cantor's function is onto.

Cantor's function is increasing, but constant almost everywhere (except on the "dust").

Cantor's function is <u>continuous.</u>

The derivative of Cantor's function is 0 almost everywhere.

7.5 THE DEVILS STAIRCASE.



Cantor's function, also called the **Devil's Staircase**, makes a continuous finite ascent (from 0 to 1) in an infinite number of steps (there are infinitely many intervals removed) while staying constant most of the time.

Check your progress

1. Prove that the Cantor set C constructed in the text is totally disconnected and perfect. In other words, given two distinct points x, y ∈ C, there is a point z /∈ C that lies in between x and y, and yet C has no isolated points. [Hint: If x, y ∈ C and |x − y| > 1/3 k , then x and y belong to two different intervals in Ck. Also, given any x ∈ C there is an end-point yk of some interval in Ck that satisfies x 6= yk and |x − yk| ≤ 1/3 k .]

7.6 LET US SUM UP

In this unit we discussed the following

Ternary Representation Of A cantor Set.

Cantor Function.

The Devils Staircase

7.7 KEYWORDS

Cantor Set-The Cantor set is set of points lying on a line segment.

7.8 QUESTIONS FOR REVIEW

2. The Cantor set C can also be described in terms of ternary expansions. (a) Every number in [0, 1] has a ternary expansion x = ∑∞ k=1 ak3 −k , where ak = 0,1 or 2. Note that this decomposition is not unique since, for example, 1/3 = ∑∞ k=2 2/3 k . Prove that x ∈ C if and only if x has a representation as above where every ak is either 0 or 2.

3.Suppose E is a given set, and On is the open set On = {x ∈ R d : d(x, E) < 1/n}.

Show: (a) If E is compact, then m(E) = limn→∞ m(On).

(b) However, the conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

Let A be the subset of [0, 1] which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find m(A).

The following deals with Gδ and Fσ sets. (a) Show that a closed set is a Gδ and an open set an Fσ. [Hint: If F is closed, consider On = {x : d(x, F) < 1/n}.]

7.9 SUGGESTED READINGS AND REFERENCES

Fundamentals of Real Analysis, S K. Berberian, Springer.

An introduction to measure theory Terence Tao

Measure Theory Authors: **Bogachev**, Vladimir I

Chovanec Ferdinand. Cantor sets. Sci. Military J. 2010

Christopher Shaver. An exploration of the cantor set. Rose-Hulman Undergraduate Mathematics Journal.

Dauben Joseph Warren, Corinthians I. Georg cantor: The battle for transfinite set theory. American Mathematical Society.

Su Francis E, et al. Devil's staircase. Math Fun Facts.

7.10 ANSWERS TO CHECK YOUR PROGRESS

1.Please check section 7.3-7.5